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Approximation results of modified Picard integral operators via regular summability methods

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Abstract. This paper focuses on applying summability methods to modified Picard integral operators and analyzing their approximation properties in exponential weighted spaces. We specifically examine Bell-type summability methods in the approximation process, noting that certain approximation results are preserved under summation. By utilizing an appropriate modulus of continuity, we determine the order of convergence of the operators, providing a deeper understanding of their behaviour in these spaces. Finally, the theoretical results are supported by numerical computation and graphical illustrations.

1. Introduction

In approximation theory, some summability methods, like subsequence matrix transforms are applied to increase the rate of convergence of sequences [8, 15]. Various summability methods have been applied in the approximation by positive linear operators [7, 9, 20]. In this work, we apply Bell type summability methods to obtain better approximation results [11]. Let $\mathcal{A} = \{A^v\} = \left\{\left[a_{jn}^v\right]\right\} \quad (j, n, v \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers and $x = \{x_n\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)_j^v$ which is defined by the $(\mathcal{A}x)_j^v = \sum_{n=1}^{\infty} a_{jn}^v x_n \quad (j, v \in \mathbb{N})$ provided that the series converges for every j, v. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{j\to\infty} (\mathcal{A}x)_j^v = L$ uniformly in $v \in \mathbb{N}$. We will denote this convergence method by $\mathcal{A} - \lim x = L$. A summability method $\mathcal{A} = \{A^v\} = \left\{\left[a_{jn}^v\right]\right\}$ is called regular if $\lim x = L$ implies $\mathcal{A} - \lim x = L$. We know that a method $\mathcal{A} = \left\{\left[a_{jn}^v\right]\right\}$ is defined as regular in [1] if and only if (a) for each $j, v \in \mathbb{N}$, $\sum_{n=1}^{\infty} \left|a_{jn}^v\right| < \infty$ and there exist integers N, M such that $\sup_{j\geq N, v\in\mathbb{N}} \sum_{n=1}^{\infty} \left|a_{jn}^v\right| \leq M$; (b) $\lim_{j\to\infty} \sum_{n=1}^{\infty} a_{jn}^v = \delta$ uniformly in v and (c) for every $n \in \mathbb{N}$, $\lim_{j\to\infty} a_{jn}^v = \delta_k$ uniformly in v with $\delta_k \equiv 0$ and $\delta \equiv 1$.

A generalized Picard operator of the following type was introduced by Agratini et al. in [3].

$$P_n^*(f; x) = \frac{\sqrt{n}}{2} \int_{\mathbb{R}} f(\alpha_n(x) + t) K_n(t) dt, \quad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{1.1}$$

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In this paper, we investigate the modified Picard integral operators defined by

$$\mathcal{P}_{j,v}(f;x) = \sum_{n=1}^{\infty} a_{jn}^{v} P_n^*(f;x) \quad (j, v \in \mathbb{N}) \tag{1.2}$$

where, the function f is selected such that the integrals are finite. In (1.2), we adopt the followings:

- $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}_{j=1}^{\infty}$ is a regular summability method.
- $P_n^*(f;x)$ is the generalized Picard integral operators defined in (1.1).
- $K_n(t) = e^{-\sqrt{n}|t|}$ is the kernel.
- $\alpha_n(x) = x - \frac{1}{2a}\ln(\frac{n}{n-4a^2})$, $n \geq n_a$ and $a > 0$.
- $n_a = \lfloor 4a^2 \rfloor + 1$, $\lfloor\cdot\rfloor$ indicating the integer part of the function or the floor function.

Note that as $a \to 0$, the following classical form of Picard integral operators are obtained

$$(P_n f)(x) = \frac{n}{2}\int_{\mathbb{R}} f(x+t)e^{-n|t|}dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N} \tag{1.3}$$

where the function f is selected so that the integral becomes finite. These operators and their modified version were widely studied by many researchers [4, 5, 17, 19]. Recently, positive linear integral operators have been introduced in several studies which preserve certain exponential functions [10, 18]. Moreover, some approximation theorems by linear and positive integral operators in weighted spaces are presented in [2]. In [12], Berdysheva, Dyn, Farkhi and Mokhov have extended the known results of the integral approximation operator for real-valued functions to general set-valued functions of bounded variation.

We have seen that $\mathcal{P}_{j,v}(f;x)$ is well defined by Proposition 3.2. In preliminaries section, we define a space containing all real valued functions whose exponential transformation is Lebesgue integrable with p^{th} power over $\mathbb{R}$, call it the exponential weighted space and denoted by $L_{p,a}(\mathbb{R})$. The approximation properties of the operators (1.2) are discussed in section 3, whereas an approximation has been established in Theorem 3.4. We compute the order of convergence of the operators in section 4 by using suitable modulus of continuity. The theoretical results are confirmed by numerical computation and graphical representation in section 5. Finally, the section 6 consists of the concluding remarks.

2. Preliminaries

For, fixed values $a > 0$ and $1 \leq p < \infty$ be fixed. Let us define

$$v_a(x) = e^{-a|x|} \quad \text{for} \quad x \in \mathbb{R}.$$

$L_{p,a}(\mathbb{R})$ be the space of all functions $f : \mathbb{R} \to \mathbb{R}$ for which $v_a f$ is Lebesgue integrable with p^{th} power over $\mathbb{R}$ defined in [16], as follows

$$L_{p,a}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \|f\|_{p,a} = \left(\int_{\mathbb{R}} |v_a(x)f(x)|^p dx\right)^{\frac{1}{p}} \right\}.$$

Let $L_{p,\rho}(\mathbb{R})$ be the linear space of measurable, p absolutely integrable functions on $\mathbb{R}$ with respect to the weight function $\rho(x)$ [14], defined as

$$L_{p,\rho}(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \middle| \|f\|_{p,\rho} = \left(\int_{\mathbb{R}} |f(x)|^p \rho(x)dx\right)^{\frac{1}{p}} < \infty \right\}.$$

3. Approximation on exponential weighted spaces

Lemma 3.1. *Let $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}$ be a nonnegative regular summability method and $\mathcal{P}_{j,v}$ be the operators given in* (1.2)*, then for each integer $k \geq 0$, $e_k = t^k$ we have*

$$\mathcal{P}_{j,v}(e_k; x) = \sum_{n=1}^{\infty} a_{jn}^{v} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{2s!}{n^s} \begin{pmatrix} k \\ 2s \end{pmatrix} \alpha_n^{k-2s}(x), \quad x \in \mathbb{R}.$$

Proof. By the definition of the operators,

$$\mathcal{P}_{j,v}(e_k; x) = \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e_k(\alpha_n(x) + t) e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} (\alpha_n(x) + t)^k e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left[\alpha_n^k(x) + \begin{pmatrix} k \\ 1 \end{pmatrix} \alpha_n^{k-1}(x) t + \begin{pmatrix} k \\ 2 \end{pmatrix} \alpha_n^{k-2}(x) t^2 + \ldots \right] e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} \alpha_n^{k-s}(x) t^s e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \sum_{s=0}^{k} \begin{pmatrix} k \\ s \end{pmatrix} \alpha_n^{k-s}(x) \int_{\mathbb{R}} t^s e^{-\sqrt{n}|t|} dt$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \frac{\sqrt{n}}{2} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \begin{pmatrix} k \\ 2s \end{pmatrix} \alpha_n^{k-2s}(x) 2 \frac{2s!}{(\sqrt{n})^{2s+1}}$$

$$= \sum_{n=1}^{\infty} a_{jn}^{v} \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} \frac{2s!}{n^s} \begin{pmatrix} k \\ 2s \end{pmatrix} \alpha_n^{k-2s}(x).$$

□

As particular cases, $j \to \infty$ we have

$(i)\ \mathcal{P}_{j,v}(e_0; x) = e_0,$

$(ii)\ \mathcal{P}_{j,v}(e_1; x) = \alpha_n(x),$

$(iii)\ \mathcal{P}_{j,v}(e_2; x) = \alpha_n^2(x) + \frac{2}{n}.$

For $f \in L_{p,a}(\mathbb{R})$ we prove that the operators $\mathcal{P}_{j,v}(f; x)$ are well defined.

Proposition 3.2. *Let $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}$ be a nonnegative regular summability method. If $f \in L_{p,a}(\mathbb{R})$, then there exists a positive constant D such that*

$$\|\mathcal{P}_{j,v}(f)\|_{p,a} \leq D \|f\|_{p,a}$$

Proof. For $f \in L_{p,a}(\mathbb{R})$,

$$\|\mathcal{P}_{j,v}(f;x)\|_{p,a} = \left(\int_{\mathbb{R}} |e^{-a|x|}\mathcal{P}_{j,v}(f;x)|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left|e^{-a|x|}\sum_{n=1}^{\infty} a_{jn}^v P_n^*(f)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left|e^{-a|x|}\sum_{n=1}^{\infty} a_{jn}^v \frac{\sqrt{n}}{2}\int_{\mathbb{R}} f(\alpha_n(x)+t)K_n(t)dt\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2}\left(\int_{\mathbb{R}} \left|e^{-a|x|}\int_{\mathbb{R}} f(\alpha_n(x)+t)e^{-\sqrt{n}|t|}dt\right|^p dx\right)^{\frac{1}{p}}.$$

By generalization of Minkowski inequality, we have

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2}\left(\int_{\mathbb{R}} e^{-\sqrt{n}|t|}\left(\int_{\mathbb{R}} \left|e^{-a|x|}f(\alpha_n(x)+t)\right|^p dx\right)^{\frac{1}{p}} dt\right)$$

using the change of variables $u = \alpha_n(x) + t$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(u)|^p e^{-a|u+\alpha_n-t|p}du\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |f(u)|^p e^{-a|u|p}du\right)^{\frac{1}{p}} e^{-a|\alpha_n-t|}e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2}\|f\|_{p,a}\int_{\mathbb{R}} e^{-a|\alpha_n-t|}e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \sqrt{n}e^{-a\alpha_n}\|f\|_{p,a}\int_{\mathbb{R}^+} e^{-(\sqrt{n}-a)t}dt$$

$$\leq D\|f\|_{p,a}$$

□

Theorem 3.3. [6] *If $f \in L_{p,a}(\mathbb{R})$ with $1 \leq p < \infty$ and $a > 0$, then we have*

$$\lim_{n\to\infty} \|P_n^*(f) - f\|_{p,a} = 0.$$

Theorem 3.4. *Let $\mathcal{A} = \left\{\left[a_{jn}^v\right]\right\}$ be a nonnegative regular summability method, then we have*

$$\lim_{j\to\infty} \|\mathcal{P}_{j,v}(f) - f\|_{p,a} = 0$$

uniformly in $v \in \mathbb{N}$, for $f \in L_{p,a}(\mathbb{R})$.

Proof. For every $f \in L_{p,a}(\mathbb{R})$,

$$\|\mathcal{P}_{j,v}(f) - f\|_{p,a} = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} P_n^*(f;x) - f \right\|_{p,a}$$

$$\leq \left\| \sum_{n=1}^{\infty} a_{jn}^{v} [P_n^*(f) - f] \right\|_{p,a} + \left\| \sum_{n=1}^{\infty} a_{jn}^{v} f - f \right\|_{p,a}$$

$$\leq I_1 + I_2$$

$$I_1 = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} [P_n^*(f) - f] \right\|_{p,a}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \sum_{n=1}^{\infty} a_{jn}^{v} [P_n^*(f) - f] \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \left(\int_{\mathbb{R}} |e^{-a|x|} [P_n^*(f) - f]|^p dx \right)^{\frac{1}{p}}$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \|P_n^*(f) - f\|_{p,a}$$

since $f \in L_{p,a}(\mathbb{R})$ using Theorem 3.3, $I_1 \to 0$.
Now,

$$I_2 = \left\| \sum_{n=1}^{\infty} a_{jn}^{v} f - f \right\|_{p,a}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \left(\sum_{n=1}^{\infty} a_{jn}^{v} f - f \right) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left| e^{-a|x|} \left(\sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right) (f) \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left| \sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right| \left(\int_{\mathbb{R}} |e^{-a|x|} (f)|^p dx \right)^{\frac{1}{p}}$$

$$\leq \left| \sum_{n=1}^{\infty} a_{jn}^{v} - 1 \right| \|f\|_{p,a}$$

as $j \to \infty$, $\quad I_2 = 0$ (uniformly in $v \in \mathbb{N}$)
The proof is completed. □

4. Order of convergence in approximation

In this section, we compute the order of convergence in Theorem 3.4. In [13], for $f \in L_{p,a}(\mathbb{R})$ and $\delta > 0$ the modulus of smoothness is defined as follows

$$\omega(f; L_{p,a}(\mathbb{R}); \delta) = \sup_{|h|\leq\delta} ||\Delta_h f(\cdot)||_{p,a}$$

where $\Delta_h f(\cdot) = f(\cdot + h) - f(\cdot)$ for $h \in \mathbb{R}$.
Let $\delta > 0$, the modulus of smoothness ω has the following properties in [6] for every $f \in L_{p,a}(\mathbb{R})$:

(*i*) $\omega(f; L_{p,a}(\mathbb{R}); \delta_1) \leq \omega(f; L_{p,a}(\mathbb{R}); \delta_2)$ for $0 \leq \delta_1 \leq \delta_2$

(*ii*) $\omega(f; L_{p,a}(\mathbb{R}); \lambda\delta) \leq (1 + \lambda)e^{a\lambda\delta}\omega(f; L_{p,a}(\mathbb{R}); \delta)$ for $\lambda,\ \delta \geq 0$

(*iii*) $\lim_{\delta\to 0} \omega(f; L_{p,a}(\mathbb{R}); \delta) = 0$

Theorem 4.1. *Let $\mathcal{A} = \left\{\left[a_{jn}^v\right]\right\}$ be a nonnegative regular summability method and $\{\delta_n\}$ be a positive null sequence, then for every $f \in L_{p,a}(\mathbb{R})$ with $1 \leq p < \infty$ and $a > 0$ we have*

$$||\mathcal{P}_{j,v}(f)(x) - f(x)||_{p,a} = O\left(\omega(f; L_{p,a}(\mathbb{R}); \alpha_n)\right) + O\left(\omega(f; L_{p,a}(\mathbb{R}); \delta_n)\right)$$

as $j \to \infty$ (uniformly in v).

Proof. For $f \in L_{p,a}(\mathbb{R})$, we have

$$||\mathcal{P}_{j,v}(f)(x) - f(x)||_{p,a} = \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\mathcal{P}_{j,v}(f; x) - f(x)\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a_{jn}^v P_n^*(f; x) - f(x)\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a_{jn}^v \frac{\sqrt{n}}{2}\int_{\mathbb{R}} (f(\alpha_n(x) + t) - f(x))e^{-\sqrt{n}|t|}dt\right)\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq \left(\int_{\mathbb{R}} \left|e^{-a|x|}\left(\sum_{n=1}^{\infty} a_{jn}^v \frac{\sqrt{n}}{2}\int_{\mathbb{R}} (f(\alpha_n(x) + t) - f(x + t) + f(x + t) - f(x))e^{-\sqrt{n}|t|}dt\right)\right|^p dx\right)^{\frac{1}{p}}.$$

Using generalization of Minkowski inequality, we get

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e^{-a|x|}(f(\alpha_n(x) + t) - f(x + t))|^p dx\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$
$$+ \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |e^{-a|x|}(f(x + t) - f(x))|^p dx\right)^{\frac{1}{p}} e^{-\sqrt{n}|t|}dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} ||f(\alpha_n(x) + t) - f(x + t))||_{p,a} e^{-\sqrt{n}|t|}dt + \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} ||f(x + t) - f(x)||_{p,a} e^{-\sqrt{n}|t|}dt$$

$$\leq \omega(f; L_{p,a}(\mathbb{R}); \alpha_n) \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e^{-\sqrt{n}|t|}dt + \sum_{n=1}^{\infty} |a_{jn}^v| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} e^{-\sqrt{n}|t|} \omega(f; L_{p,a}(\mathbb{R}); |t|)dt$$

using the properties of modulus of smoothness (*ii*) we have

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \int_{\mathbb{R}} (1 + \sqrt{n}|t|) e^{a|t|} \omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) e^{-\sqrt{n}|t|} dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}| \frac{\sqrt{n}}{2} \omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) \int_{\mathbb{R}} (1 + \sqrt{n}|t|) e^{a|t|} e^{-\sqrt{n}|t|} dt$$

$$\leq \sum_{n=1}^{\infty} |a_{jn}^{v}| \omega(f; L_{p,a}(\mathbb{R}); \alpha_n) + \sum_{n=1}^{\infty} |a_{jn}^{v}| \omega\left(f; L_{p,a}(\mathbb{R}); \frac{1}{\sqrt{n}}\right) \frac{2n}{(\sqrt{n} - a)^2}$$

$$\leq O(\omega(f; L_{p,a}(\mathbb{R}); \alpha_n)) + O(\omega(f; L_{p,a}(\mathbb{R}); \delta_n))$$

(uniformly in v). □

Theorem 4.2. *Let $\mathcal{A} = \left\{\left[a_{jn}^{v}\right]\right\}$ be a nonnegative regular summability method, then for every $f \in L_{p,a}(\mathbb{R})$ with $1 \leq p < \infty$ and $\delta, a > 0$ we have*

$$\omega(\mathcal{P}_{j,v}(f); L_{p,a}(\mathbb{R}); \delta) \leq D\omega(f; L_{p,a}(\mathbb{R}); \delta)$$

Proof. For $h \in \mathbb{R}$

$$\|\mathcal{P}_{j,v}(f)(\cdot + h) - f(\cdot)\|_{p,a} \leq \|\mathcal{P}_{j,v}(f(\cdot + h) - f(\cdot))\|_{p,a}$$

$$\leq D\|f(\cdot + h) - f(\cdot)\|_{p,a} \quad \text{(using Proposition 3.2)}$$

$$\leq D\omega(f; L_{p,a}(\mathbb{R}); \delta).$$

□

5. Numerical computations and Graphical illustrations

In this section, we approximate to the function $f(x) = |x|$ by means of $\mathcal{P}_{j,v}(f; x)$ in (1.2), where we consider the Cesàro matrix summability $\mathcal{A} = \{C_1\} = \{[c_{jn}]\}$ given by

$$c_{jn} = \begin{cases} \frac{1}{j}, & \text{if } 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Approximation errors are listed in Table 1.

j	Error of approximation
7	≈ 0.2736
16	≈ 0.2305
51	≈ 0.1438
104	≈ 0.0908
209	≈ 0.0533
500	≈ 0.0342

Table 1: Upper bound of error for certain values of j

We also show graphical illustrations indicating the approximation to a function f by means of $\mathcal{P}_{j,v}(f; x)$ operators in Figure 1. We have seen in Figure 1 that when the value of j increases the error decreases and $\mathcal{P}_{j,v}$ gives a better approximation of the original function $f(x) = |x|$.

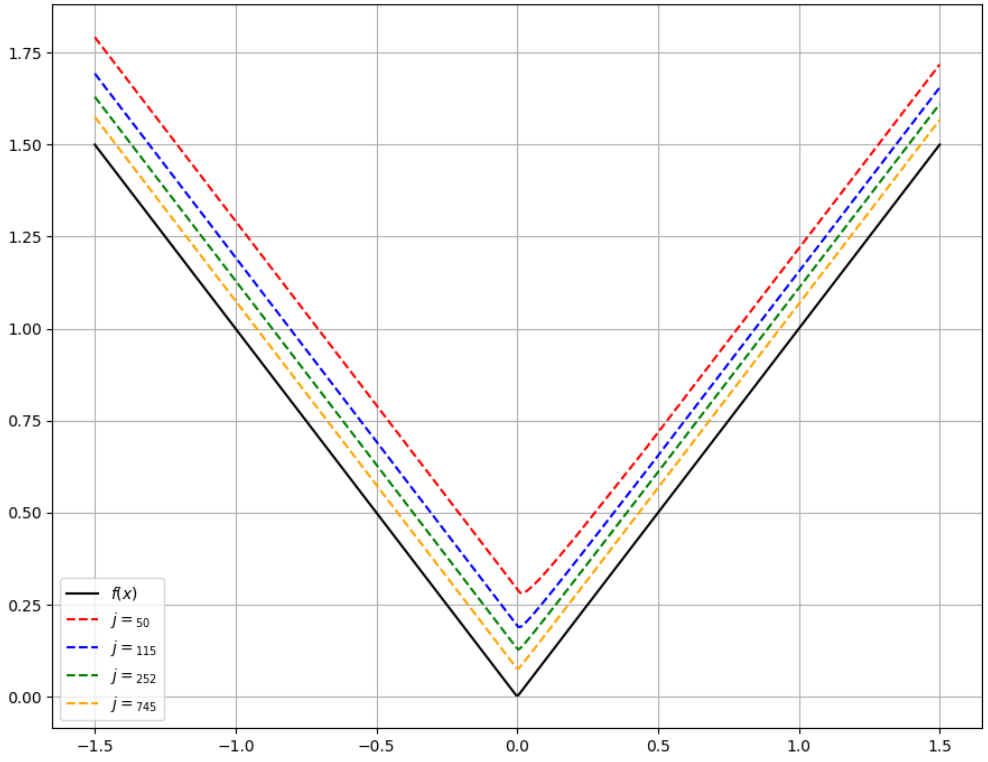


Figure 1: Approximation to $f(x) = |x|$ by means of $\mathcal{P}_{j,v}$ for the values of $j = 50, 115, 252, 745$

6. Conclusion

In this work, we have studied some approximation properties of the modified Picard integral operators in exponential weighted spaces. The regular summability methods have been applied to derive more generalized results compared to those found in existing literature. By employing the weighted modulus of continuity, we have established an upper bound for the order of convergence of the modified Picard integral operators in exponential weighted spaces. It is also observed that some results in approximation theory remain unchanged under summability methods. Using an exponential weighted modulus of continuity, the result regarding global smoothness preservation properties have been proved in Theorem 4.2. Some graphical illustrations and numerical example are also provided for the confirmation of theoretical outcomes.

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Approximation of Mellin convolution-type nonlinear integral operators in variable bounded variation spaces

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Abstract

In this paper, we investigate approximation properties using a family of Mellin convolution-type integral operators within the framework of variable bounded variation spaces with the help of summability methods. In this approximation, we use Bell-type summability methods. We also investigate the impact of regular summability methods on the approximation properties of these operators. Moreover, we examine the rate of approximation for certain functions belonging to a suitable Lipschitz class.

Keywords: Mellin Convolution type operators; Summability method; Bounded variation spaces with variable exponent; Modulus of smoothness; Rate of approximation

Mathematics Subject Classification: 26B30; 26A45; 41A25; 41A35

1 Introduction

Integral operators of Mellin convolution type are commonly applied in approximation theory [1–5], where some of them have linear structures [6, 7] and others have nonlinear structures [8, 9]. Apart from approximation theory, Mellin type operators have numerous applications in various fields including optical physics, signal analysis and engineering [10–13]. Our study is basically motivated by the work of Angeloni and Vinti [14, 15], which examine the approximation properties of nonlinear integral operators.

The primary goal of using summability methods has always been to make a nonconvergent sequence to a convergent sequence. If the sequence of positive linear operators fails to converge, then the matrix summability methods become more beneficial [16, 17]. Matrix summability methods have been shown to be highly effective in summing sequences of nonlinear integral operators [18, 19]. There are numerous summability methods such as Cesàro method, almost convergence and order summability method. We consider Bell-type summability method [20], which is a general method that includes all others.

It is known that the concept of bounded variation theory was first initiated by C. Jordan in 1880 [21]. Later many mathematicians, such as N. Wiener, L.C. Young, R. E. Love, W.

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Orlicz, J. Musielak, and L. Tonelli expanded the Jordan variation in different dimensions [22–25]. The variable bounded variation space is a generalization of classical bounded variation spaces in the sense of Jordan [21]. Function spaces with variable exponents are currently dynamic area of research. Castillo et al. [26] specifically introduced variable bounded variation spaces, referencing the concept defined by Wiener in [27]. Research into variable bounded variation spaces is advancing quickly due to their intrinsic interest, and also for the important applications in various domains including digital image processing [28, 29], in the modeling of electrorheological fluids [30] as well as thermorheological fluids [31], and in differential equations with non-standard growth [32]. These aforementioned applications are mainly in Lebesgue and Sobolev space with variable integrability which has been challenging topic in last three decades. Such spaces, introduced by Orlicz [33] and then developed by Nakano [34] are a generalization of classical Lebesgue spaces, the basic idea is that the constant exponent p of the L^p spaces is replaced by variable function $p(\cdot)$. Such spaces share with the classical L^p spaces, several properties, but nevertheless they also present some significant differences. As an example, the translation operator applied to a function belonging to a variable Lebesgue spaces do not belong to the same spaces, as it holds in L^p spaces [35], and of course the same happens in $BV^{p(\cdot)}$ spaces. Another delicate point is about the additivity of the variation on intervals: analogously to what happens in the case of Musielak-Orlicz φ-variation [24], the classical additivity property on intervals is replaced by suitable inequalities ([27] see Proposition 3.2). These facts make the problem of convergence in variable variation much more delicate with respect to working with the classical variation.

2 Basic definitions and preliminaries

The following preliminaries of variable bounded variation spaces in the sense of Wiener and summability theory are given below:

Definition 2.1 ([26]) An admissible function $p(\cdot)$, is a measurable function $p : \mathbb{R} \to [1, +\infty)$, such that

$$1 \leq p_- = \inf_{x\in\mathbb{R}} p(x) \leq p_+ = \sup_{x\in\mathbb{R}} p(x) < \infty.$$

Definition 2.2 ([27]) Let $f : \mathbb{R}_+ \to \mathbb{R}$. The $p(\cdot)$-variation in the Wiener's sense of f is defined on $\mathbb{R}$ as

$$V^{p(\cdot)}[f] = \sup_{\Pi^*_\delta} \sum_{i=1}^{n} \left| f(t_i) - f(t_{i-1}) \right|^{p(x_{i-1})},$$

where Π^*_δ is a tagged sequence, i.e., an increasing sequence $t_0 < t_1 < t_2 < \cdots < t_n$, together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ subject to the condition $t_i \leq x_i \leq t_{i+1}$, $\forall\, i = 0, \ldots, n-1$.

Definition 2.3 ([27]) By

$$BV^{p(\cdot)}(\mathbb{R}_+) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} : V^{p(\cdot)}[\lambda f] < \infty, \text{ for some } \lambda > 0 \right\},$$

we denote the space of functions of bounded $p(\cdot)$-variation on $\mathbb{R}_+$. Also $BV^{p(\cdot)}(\mathbb{R}_+)$ is defined by means of the norm

$$\|f\|_{BV^{p(\cdot)}(\mathbb{R}_+)} = \inf\left\{\lambda > 0 : V^{p(\cdot)}\left(\frac{f}{\lambda}, \mathbb{R}_+\right) \leq 1\right\},$$

i.e., as the space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ for which $\|f\|_{BV^{p(\cdot)}(\mathbb{R}_+)} < +\infty$. It can be seen that [27], $\|f\|_{BV^{p(\cdot)}(\mathbb{R}^+)}$ is a Luxemburg norm. Equipped the space $BV^{p(\cdot)}(\mathbb{R}^+)$ with this norm we have a Banach space.

Definition 2.4 ([27]) A family of functions $(f_w)_{w>0} \subset BV^{p(\cdot)}(\mathbb{R}_+)$ is convergent in variation (modular convergent) to $f \in BV^{p(\cdot)}(\mathbb{R}_+)$ if there exists $\lambda > 0$, such that

$$V^{p(\cdot)}\left[\lambda\left(f_w - f\right), \mathbb{R}_+\right] \to 0, \quad \text{as } w \to +\infty.$$

Besides modular convergence, that is the notion of convergence that we will use in this paper, the norm $\|f\|_{BV^{p(\cdot)}(\mathbb{R}_+)}$ induces the usual norm convergence. In general, norm convergence is stronger than modular convergence. We recall that norm convergence is equivalent to modular convergence in case of $p_+ < +\infty$ (i.e., $\|f_w - f\|_{BV^{p(\cdot)}(\mathbb{R}_+)} \to 0$ as $w \to +\infty$ is equivalent to $V^{p(\cdot)}\left[\lambda\left(f_w - f\right), \mathbb{R}_+\right] \to 0$, as $w \to +\infty$ for every $\lambda > 0$).

Definition 2.5 ([26]) By modulus of $p(\cdot)$-continuity of the function $f : \mathbb{R}_+ \to \mathbb{R}$ we mean

$$\omega_\delta^{p(\cdot)}(f) = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} \left|f(t_i) - f(t_{i-1})\right|^{p(x_i - 1)},$$

where Π_δ^* is a tagged partition of $\mathbb{R}_+$ with mesh not greater than δ.

By $AC^{p(\cdot)}(\mathbb{R}_+)$, we will denote the space of all the absolutely $p(\cdot)$-continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ and defined as

$$AC^{p(\cdot)}(\mathbb{R}_+) = \left\{\text{for some } \lambda > 0 : f \in BV^{p(\cdot)}(\mathbb{R}_+) \text{ and } \lim_{\delta \to 0} \omega_\delta^{p(\cdot)}(\lambda f) = 0\right\}.$$

In particular case $p(\cdot) = 1$, the above definition reduces (for $\lambda = 1$) to the classical absolute continuity, expressed in terms of convergence of the modulus of continuity, i.e., $\lim_{\delta \to 0} \omega_\delta^1 = 0$. The space of absolutely continuous functions on $\mathbb{R}_+$ is generally represented by $AC(\mathbb{R}_+)$. We observe that $AC(\mathbb{R}_+) \subset AC^{p(\cdot)}(\mathbb{R}_+)$. Indeed, first of all recall that $p(\cdot) \geq 1$ implies $BV(\mathbb{R}_+) \subset BV^{p(\cdot)}(\mathbb{R}_+)$.

Proposition 2.6 ([26]) *If $p(\cdot) \leq q(\cdot)$, then for any $\lambda > 0$,*

$$V^{p(\cdot)}\left[\lambda f\right] \geq V^{q(\cdot)}\left[\lambda f\right]$$

and therefore, $BV^{p(\cdot)}(\mathbb{R}_+) \subset BV^{q(\cdot)}(\mathbb{R}_+)$.

Proposition 2.7 ([27]) *If* $f_1,\dots,f_m \in BV^{p(\cdot)}(\mathbb{R}_+)$, $m \in \mathbb{N}$, *then* $f_1+\cdots+f_m \in BV^{p(\cdot)}(\mathbb{R}_+)$ *and, for some* $\lambda > 0$

$$V^{p(\cdot)}\left[\sum_{i=1}^{m}\lambda f_i\right] \le m^{p_+-1}\sum_{i=1}^{m} V^{p(\cdot)}\left[\lambda f_i\right].$$

Definition 2.8 ([36]) Let $\mathcal{A} = \{A^\nu\} = \left\{\left[a_{nw}^\nu\right]\right\}$ $(n,w,\nu \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Let $x = \{x_w\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)_n^\nu$ which is defined by the $(\mathcal{A}x)_n^\nu = \sum_{w=1}^{\infty} a_{nw}^\nu x_w$ $(n,\nu \in \mathbb{N})$ provided that the series converges for every n, ν. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{n\to+\infty}(\mathcal{A}x)_n^\nu = L$ uniformly in $\nu \in \mathbb{N}$. In this case we write $\mathcal{A}-\lim x = L$.

Since the summability method is generated by a family of infinite matrices $\mathcal{A} = \{A^\nu\}$ $(\nu \in \mathbb{N})$, instead of a single infinite matrix, it is classified as a Bell-type summability method.

Definition 2.9 ([36]) A summability method $\mathcal{A} = \{A^\nu\} = \left\{\left[a_{nw}^\nu\right]\right\}$ is called regular if $\lim x_w = L$ implies $\mathcal{A}-\lim x = L$.

Definition 2.10 ([37]) A method $\mathcal{A} = \left\{\left[a_{nw}^\nu\right]\right\}$ is regular if and only if the following conditions hold:

(a) for each $n,\nu \in \mathbb{N}$, $\sum_{w=1}^{+\infty}\left|a_{nw}^\nu\right| < +\infty$ and there exist integers N, M such that $\sup_{n\ge N,\nu\in\mathbb{N}}\sum_{w=1}^{+\infty}\left|a_{nw}^\nu\right| \le M$,
(b) $\lim_{n\to+\infty}\sum_{w=1}^{+\infty} a_{nw}^\nu = \delta$ uniformly in ν,
(c) for every $n \in \mathbb{N}$, $\lim_{n\to+\infty} a_{nw}^\nu = \delta_k$ uniformly in ν with $\delta_k \equiv 0$ and $\delta \equiv 1$.

In this work, we study the approximation properties of the nonlinear integral operators of Mellin convolution-type [38]

$$T_w(f;s) = \int_0^{+\infty} K_w(t,f(st))\frac{dt}{t}. \tag{2.1}$$

Here, the kernel $K_w : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a family of measurable functions such that $K_w(s,\ t) = L_w(s)\,H_w(t)$ for every $s \in \mathbb{R}_+$. $L_w : \mathbb{R}_+ \to \mathbb{R}$ is Haar measurable on $\mathbb{R}_+$ and $L_w \in L_\mu^1(\mathbb{R}_+)$, where μ corresponds to Haar measure on $\mathbb{R}_+$ ($\mu(A) = \int_A d\mu = \int_A t^{-1}dt$, where A is a Borel subset of $\mathbb{R}_+$) and

$$L_\mu^1(\mathbb{R}_+) = \left\{f : \mathbb{R}_+ \to \mathbb{R} : \|f\|_{L_\mu^1} = \int_0^{+\infty}\left|f(t)\right|\frac{dt}{t} < +\infty\right\}$$

and $\{H_w\}$ is a family of functions such that $H_w(0) = 0$ having the Lipschitz property, i.e., there exists a constant $C > 0$ such that $\left|H_w(x) - H_w(y)\right| \le C\left|x-y\right|$ for every $x,y \in \mathbb{R}$ and $w \in \mathbb{N}$.

$L_\mu^\infty(\mathbb{R}_+)$ denotes the space of all essentially bounded functions on $\mathbb{R}_+$ with respect to Haar measure and $\|\cdot\|_{L_\mu^\infty}$ denotes the associated norm.

Now using matrix summability methods $\mathcal{A} = \{A^\nu\} = \left\{\left[a_{nw}^\nu\right]\right\}$ $(n,w,\nu \in \mathbb{N})$, our approximation operators take the following form:

$$\mathcal{T}_{n,\nu}(f;s) = \sum_{w=1}^{+\infty} a_{nw}^\nu \int_0^{+\infty} K_w(t,f(st))\frac{dt}{t};\ n,\nu \in \mathbb{N}. \tag{2.2}$$

We obtain from (2.1) and (2.2) that

$$\mathcal{T}_{n,\nu}(f;s) = \sum_{w=1}^{+\infty} a_{nw}^{\nu} T_w(f;s). \tag{2.3}$$

The modulus of smoothness, in the context of $BV^{p(\cdot)}(\mathbb{R}_+)$ spaces, for $f : \mathbb{R}_+ \to \mathbb{R}$, defined as

$$\omega^{p(\cdot)}(f,\delta) = \sup_{|t|\le\delta} V^{p(\cdot)}\left[\tau_t f - f\right], \delta > 0$$

where $\tau_t f(s) = f(st)$ for every $s,\ t \in \mathbb{R}_+$.

3 Approximation results through Mellin convolution type operators

In this section, we shall study the approximation properties in $BV^{p(\cdot)}(\mathbb{R}_+)$ of operator (2.2), for the main approximation theorem, we also need the following general assumptions on L_w:

(i) There exists a constant $D > 0$, such that $sup_{w\in\mathbb{N}} \|L_w\|_{L_\mu^1} \le D < +\infty$.
(ii) $\mathcal{A} - \lim\left(\int_0^{+\infty} L_w(t)\frac{dt}{t}\right) = 1$.
(iii) for every fixed $\delta > 0$, $\mathcal{A} - \lim\left(\int_{|1-t|\ge\delta} |L_w(t)|\,\frac{dt}{t}\right) = 0$.
(iv) Let $G_w(u) = H_w(u) - u$, $u \in \mathbb{R}$, for every $\gamma > 0$, there exists a $\lambda > 0$, such that $\lim_{w\to+\infty} \frac{V^{p(\cdot)}[\lambda G_w]}{\gamma m(J)} = 0$ uniformly with respect to every bounded interval $J \subseteq \mathbb{R}$, which means that for every $\varepsilon > 0$, there exists $w_0 \in \mathbb{N}$, such that $\forall\ w \ge w_0$, $V^{p(\cdot)}[\lambda G_w] < \varepsilon\gamma m(J)$, where $m(J)$ is the length of the interval J.

Lemma 3.1 *Let $\mathcal{A} = \{[a_{nw}^{\nu}]\}$ be a nonnegative regular summability method. Then for all $f \in L_\mu^\infty(\mathbb{R}_+)$, $\mathcal{T}_{n,\nu}(f;s) < +\infty$ $\forall s \in \mathbb{R}_+$ and assume that condition (i) holds, then $\mathcal{T}_{n,\nu}(f) \in L_\mu^1(\mathbb{R}_+)$ for every $f \in L_\mu^1(\mathbb{R}_+)$.*

Proof For $s \in \mathbb{R}_+$, from (2.2) using Hölder inequality we have

$$\begin{aligned}\left|\mathcal{T}_{n,\nu}(f;s)\right| &\le \sum_{w=1}^{+\infty} a_{nw}^{\nu} \int_0^{+\infty} |L_w(t)|\left|H_w(f(st))\right|\frac{dt}{t} \le C\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_0^{+\infty}|L_w(t)|\left|f(st)\right|\frac{dt}{t}\\ &\le C\sum_{w=1}^{+\infty} a_{nw}^{\nu}\|L_w\|_{L_\mu^1}\|f\|_{L_\mu^\infty} \le CD\|f\|_{L_\mu^\infty}\sum_{w=1}^{+\infty} a_{nw}^{\nu} \le +\infty.\end{aligned}$$

Considering the condition (i), then by the Fubini-Tonelli theorem we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+}\left|\mathcal{T}_{n,\nu}(f;s)\right|\frac{ds}{s} &\le \int_0^{+\infty}\left(\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_0^{+\infty}|L_w(t)|\left|H_w(f(st))\right|\frac{dt}{t}\right)\frac{ds}{s}\\ &\le C\int_0^{+\infty}\sum_{w=1}^{+\infty} a_{nw}^{\nu}|L_w(t)|\left(\int_0^{+\infty}\left|f(st)\right|\frac{ds}{s}\right)\frac{dt}{t}\\ &\le C\sum_{w=1}^{+\infty} a_{nw}^{\nu}\|L_w\|_{L_\mu^1}\|f\|_{L_\mu^1} \le CD\|f\|_{L_\mu^1}\sum_{w=1}^{+\infty} a_{nw}^{\nu} < +\infty.\end{aligned}$$

Thus, $\mathcal{T}_{n,\nu}(f) \in L_\mu^1(\mathbb{R}_+)$. □

Theorem 3.2 *Let* $\mathcal{A} = \{[a_{nw}^{v}]\}$ *be a nonnegative regular summability method. If* $f \in BV^{p(\cdot)}(\mathbb{R}_+)$ *and condition* (*i*) *is satisfied, then there exists a* $\eta > 0$, *such that*

$$V^{p_+/p_-p(\cdot)}[\eta \mathcal{T}_{n,v}(f)] \leq V^{p(\cdot)}[\lambda f],$$

$\mathcal{T}_{n,v}$ *maps from* $BV^{p(\cdot)}(\mathbb{R}_+)$ *into* $BV^{p_+/p_-p(\cdot)}(\mathbb{R}_+)$, *where* $\lambda > 0$ *is such that* $V^{p(\cdot)}[\lambda f] < +\infty$.

Proof Let $s_0 < s_1 < \cdots < s_m, x_0 < x_1 < \cdots < x_{m-1}$ be a tagged sequence in $\mathbb{R}_+$. Then, for $\eta > 0$, we have

$$\begin{aligned} U &= \sum_{i=1}^{m} \left|\eta\left[\mathcal{T}_{n,v}(f;s_i) - \mathcal{T}_{n,v}(f;s_{i-1})\right]\right|^{p_+/p_-p(x_{i-1})} \\ &= \sum_{i=1}^{m} \left| \eta \left[\sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} L_w(t) H_w(f(s_i t)) \frac{dt}{t} \right.\right. \\ &\qquad \left.\left. - \sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} L_w(t) H_w(f(s_{i-1} t)) \frac{dt}{t} \right] \right|^{p_+/p_-p(x_{i-1})}. \end{aligned}$$

Now, by the convexity of the function $u^{p_+/p_-p(\cdot)}$, $u \geq 0$, condition (i) and Jensen's inequality

$$\begin{aligned} &\leq \frac{1}{D} \sum_{i=1}^{m} \left| \eta D \left[\sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \left| H_w\left(f(s_i t)\right) - H_w(f(s_{i-1} t)) \right| \frac{dt}{t} \right] \right|^{p_+/p_-p(x_{i-1})} \\ &\leq \frac{C}{D} \sum_{i=1}^{m} \left| \eta D \left[\sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \left| f(s_i t) - f(s_{i-1} t) \right| \frac{dt}{t} \right] \right|^{p_+/p_-p(\cdot)} \\ &\leq \frac{C}{D} \sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \sum_{i=1}^{m} \left[\eta D \left| f(s_i t) - f(s_{i-1} t) \right| \right]^{p_+/p_-p(\cdot)} \frac{dt}{t} \\ &\leq \frac{C}{D} \sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \, V^{p_+/p_-p(\cdot)}[\eta D f(\cdot t)] \frac{dt}{t}. \end{aligned}$$

Since, for every $t \in \mathbb{R}_+$, $p_+/p_-p(\cdot) \geq p(\cdot)$, then by Proposition 2.6

$$U \leq \frac{C}{D} \sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \, V^{p(\cdot)}\left[\eta D f(\cdot t)\right] \frac{dt}{t}.$$

Therefore, if $0 < \eta < \frac{\lambda}{D}$, passing to the supremum over all the possible tagged sequences in $\mathbb{R}_+$, we have

$$U \leq \frac{C}{D} V^{p(\cdot)}\left[\lambda f(\cdot t)\right] \sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} |L_w(t)| \frac{dt}{t}.$$

Taking, $V^{p(\cdot)}\left[\lambda f(\cdot t)\right] = V^{p(\cdot)}[\lambda f]$, from (i) and (a) we finally conclude that

$$V^{p_+/p_-p(\cdot)}[\eta \mathcal{T}_{n,v}(f)] \leq V^{p(\cdot)}[\lambda f]. \qquad \square$$

Remark 3.3

Case I: If $p(\cdot) = 1$, the variable bounded variation reduces to the classical Jordan variation and the estimate of Theorem 3.2 becomes

$$V[\mathcal{T}_{n,v}(f)] \leq V[f]$$

that is the variation diminishing property for the Mellin convolution type integral operators [3].

Case II: If $p(\cdot) = p$, $p > 1$ the variable bounded variation spaces coincide with the Wiener p-variation and the estimate of Theorem 3.2 reduces to the variation diminishing property for the Wiener p-variation ([24] for generalization of such result in the case of the Musielak-Orlicz-φ-variation).

We have seen that in all these cases with classical variation, the Mellin convolution type operators map the space of functions of bounded variation in the sense of Jordan, Wiener to itself, while they map $BV^{p(\cdot)}(\mathbb{R}_+)$ into $BV^{p_+/p_-p(\cdot)}(\mathbb{R}_+)$. This is common in the setting of variable exponent spaces as well as in variable Lebesgue spaces, that are not translation invariant [35].

Theorem 3.4 *Let $\mathcal{A} = \{[a^v_{nw}]\}$ be a nonnegative regular summability method. Assume conditions (i)-(iv) hold. Then for every $f \in AC^{p(\cdot)}(\mathbb{R}_+)$ there exists $\lambda > 0$, such that*

$$\lim_{n\to\infty} V^{p_+^2/p_-^2 p(\cdot)}[\lambda(\mathcal{T}_{n,v}(f) - f)] = 0$$

uniformly in $v \in \mathbb{N}$.

Proof For a fixed tagged sequence $s_0 < s_1 < \cdots < s_m$, $x_0 < x_1 < \cdots < x_{m-1}$ in $\mathbb{R}_+$ and $\lambda > 0$, then we get the following inequality

$$\begin{aligned}
U &= \sum_{i=1}^{m} \left|\lambda\left[\mathcal{T}_{n,v}(f;s_i) - f(s_i) - \mathcal{T}_{n,v}(f;s_{i-1}) + f(s_{i-1})\right]\right|^{p_+^2/p_-^2 p(x_{i-1})} \\
&= \sum_{i=1}^{m} \left| \lambda \left[\sum_{w=1}^{+\infty} a^v_{nw} \int_0^{+\infty} L_w(t) \left\{ H_w\left(f(s_i t)\right) - f(s_i t) - H_w\left(f(s_{i-1} t)\right) + f(s_{i-1} t) \right\} \frac{dt}{t} \right.\right. \\
&+ \sum_{w=1}^{+\infty} a^v_{nw} \int_0^{+\infty} L_w(t) \left\{ f(s_i t) - f(s_i) - f(s_{i-1} t) + f(s_{i-1}) \right\} \frac{dt}{t} \\
&+ \left.\left. \left(\sum_{w=1}^{+\infty} a^v_{nw} \int_0^{+\infty} L_w(t) \frac{dt}{t} - 1 \right) \left\{ f(s_i) - f(s_{i-1}) \right\} \right] \right|^{p_+^2/p_-^2 p(x_{i-1})} \\
&\leq \frac{1}{3} \sum_{i=1}^{m} \left[3\lambda \sum_{w=1}^{+\infty} a^v_{nw} \int_0^{+\infty} |L_w(t)| \left| H_w\left(f(s_i t)\right) - f(s_i t) - H_w\left(f(s_{i-1} t)\right) \right.\right. \\
&\left.\left. + f(s_{i-1} t) \right| \frac{dt}{t} \right]^{p_+^2/p_-^2 p(x_{i-1})} \\
&+ \frac{1}{3} \sum_{i=1}^{m} \left[3\lambda \sum_{w=1}^{+\infty} a^v_{nw} \int_0^{+\infty} |L_w(t)| \left| f(s_i t) - f(s_i) - f(s_{i-1} t) + f(s_{i-1}) \right| \frac{dt}{t} \right]^{p_+^2/p_-^2 p(x_{i-1})}
\end{aligned}$$

$$+\frac{1}{3}\sum_{i=1}^{m}\left[3\lambda\left|\sum_{w=1}^{+\infty}a_{nw}^{v}\int_{0}^{+\infty}L_w(t)\,\frac{dt}{t}-1\right|\left|f(s_i)-f(s_{i-1})\right|\right]^{p_+^2/p_-^2p(x_{i-1})}$$
$$=U_1+U_2+U_3.$$

Now,

$$U_1=\frac{1}{3}\sum_{i=1}^{m}\left[3\lambda\sum_{w=1}^{+\infty}a_{nw}^{v}\int_{0}^{+\infty}|L_w(t)|\left|H_w\left(f(s_it)\right)-f(s_it)-H_w\left(f(s_{i-1}t)\right)\right.\right.$$
$$\left.\left.+f(s_{i-1}t)\right|\frac{dt}{t}\right]^{p_+^2/p_-^2p(x_{i-1})}$$

using Jensen's inequality in U_1 with (i) and (a), we get

$$U_1=\frac{1}{3M}\sum_{i=1}^{m}\sum_{w=1}^{+\infty}a_{nw}^{v}\left[3M\lambda\int_{0}^{+\infty}|L_w(t)|\left|H_w\left(f(s_it)\right)-f(s_it)-H_w\left(f(s_{i-1}t)\right)\right.\right.$$
$$\left.\left.-f(s_{i-1}t)\right|\right]^{p_+^2/p_-^2p(x_{i-1})}\frac{dt}{t}$$
$$\leq\frac{1}{3MD}\sum_{w=1}^{+\infty}a_{nw}^{v}\int_{0}^{+\infty}|L_w(t)|\sum_{i=1}^{m}\left[3MD\lambda\left|H_w\left(f(s_it)\right)-f(s_it)-H_w\left(f(s_{i-1}t)\right)\right.\right.$$
$$\left.\left.-f(s_{i-1}t)\right|\right]^{p_+^2/p_-^2p(x_{i-1})}\frac{dt}{t}$$
$$\leq\frac{1}{3MD}\sum_{w=1}^{+\infty}a_{nw}^{v}\int_{0}^{+\infty}|L_w(t)|\,V^{p_+^2/p_-^2p(x_{i-1})}[3MD\lambda(H_w\left(f(\cdot t)\right)-f(\cdot t))]\frac{dt}{t}$$
$$\leq\frac{1}{3M}\sum_{w=1}^{+\infty}a_{nw}^{v}V^{p_+^2/p_-^2p(x_{i-1})}[3MD\lambda(H_w\left(f\right)-f)].$$

Now considering (iv) in Lemma 1 in [15], for a given $\varepsilon>0$ there exists $w_0\in\mathbb{N}$ such that $V^{p_+^2/p_-^2p(x_{i-1})}[3MD\lambda(H_w\left(f\right)-f)]<\varepsilon$ for every $w\geq w_0$ holds for some $\lambda>0$ and $\varepsilon<1$. Then, dividing the sum as the following, we get

$$U_1\leq\frac{1}{3M}\left(\sum_{w=1}^{w_0-1}a_{nw}^{v}V^{p_+^2/p_-^2p(x_{i-1})}\left[3MD\lambda\left(H_w\left(f\right)-f\right)\right]\right.$$
$$\left.+\sum_{w=w_0}^{+\infty}a_{nw}^{v}V^{p_+^2/p_-^2p(x_{i-1})}[3MD\lambda(H_w\left(f\right)-f)]\right)$$
$$=U_1^1+U_1^2.$$

Since $\mathcal{A}$ is regular and $w\in\{1,\,2,\,\ldots,w_0-1\}$ in U_1^1, we can find a number w_0, only depending on ε, such that for all $w\geq w_0$,

$$U_1^1<\frac{P}{3M}(w_0-1)\varepsilon$$

holds, where $P=\max_{w\in\{1,\,2,\,\ldots,w_0-1\}}\left\{V^{p_+^2/p_-^2p(x_{i-1})}[3MD\lambda(H_w\left(f\right)-f)]\right\}$.

From the regularity property (a), we get

$$U_1^2 \leq \frac{\varepsilon}{3}.$$

For U_2, by the Jensen's inequality we observe that

$$\begin{aligned}
U_2 &\leq \frac{1}{3M}\sum_{i=1}^{m}\sum_{w=1}^{+\infty} a_{nw}^{v}\left[3M\lambda\int_0^{+\infty}|L_w(t)|\left|f(s_i t)-f(s_i)-f(s_{i-1}t)\right.\right.\\
&\quad \left.\left.+f(s_{i-1})\right|\right]^{p_+^2/p_-^2 p(x_{i-1})}\frac{dt}{t}\\
&\leq \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_0^{+\infty}|L_w(t)|\sum_{i=1}^{m}\left[3MD\lambda\left|f(s_i t)-f(s_i)-f(s_{i-1}t)\right.\right.\\
&\quad \left.\left.+f(s_{i-1})\right|\right]^{p_+^2/p_-^2 p(x_{i-1})}\frac{dt}{t}\\
&\leq \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_0^{+\infty}|L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}[3MD\lambda(\tau_t f-f)]\frac{dt}{t}
\end{aligned}$$

where $\tau_t f(s) = f(st)$ $(s,t\in\mathbb{R}_+)$ denotes the dilation operator. By Theorem 4.5 in [27], for every $\varepsilon>0$, there exists $\overline{\lambda}>0$, such that,

$$V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f-f\right)\right]\leq\varepsilon \tag{3.1}$$

whenever $|1-t|<\delta$ and $\lambda\leq\frac{\overline{\lambda}}{3MD}$.

$$\begin{aligned}
U_2 &\leq \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|<\delta}|L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f-f\right)\right]\frac{dt}{t}\\
&\quad+\frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|\geq\delta}|L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}[3MD\lambda(\tau_t f-f)]\frac{dt}{t}\\
U_2 &\leq U_2^1+U_2^2.
\end{aligned}$$

From (3.1), (i) and (a), we get

$$U_2^1\leq\frac{\varepsilon}{3}.$$

As $V^{p(\cdot)}\left[3MD\lambda\tau_t f\right]=V^{p(\cdot)}\left[3MD\lambda f\right]$, then by Proposition 2.6 and Proposition 2.7 we get

$$\begin{aligned}
&V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f-f\right)\right]\\
&\leq V^{p_+/p_- p(\cdot)}[3MD\lambda(\tau_t f-f)]\\
&\leq 2^{p_+-1}\left\{V^{p(\cdot)}\left[3MD\lambda\tau_t f\right]+V^{p(\cdot)}\left[3MD\lambda f\right]\right\} \qquad (3.2)\\
&\leq 2^{p_+}V^{p(\cdot)}\left[3MD\lambda f\right]\\
&\leq 2^{p_+}V^{p(\cdot)}\left[\overline{\lambda}f\right].
\end{aligned}$$

For $0 < \lambda < \frac{\overline{\lambda}}{3MD}$, where $\overline{\lambda}$ is such that $V^{p(\cdot)}\left[\overline{\lambda}f\right] < +\infty$ and by condition (iii), there exists $\overline{w} > 0$ such that

$$\sum_{w=1}^{+\infty} a_{nw}^{v} \int_{|1-t|\geq\delta} |L_w(t)| \frac{dt}{t} < \frac{\varepsilon}{M^{-1}D^{-1}2^{p_+}V^{p(\cdot)}[\overline{\lambda}f]}$$

for every $w \geq \overline{w}$, and so

$$U_2^2 < \frac{\varepsilon}{3}.$$

Now, from (ii), for every $0 < \varepsilon < 1$, there exists a number n_0, such that, for all $n \geq n_0$

$$\left|\sum_{w=1}^{+\infty} a_{nw}^{v} \int_0^{+\infty} L_w(t) \frac{dt}{t} - 1\right| < \varepsilon < 1.$$

Hence

$$\begin{aligned} U_3 &\leq \frac{1}{3}\sum_{i=1}^{m}\left[3\lambda\left|\sum_{w=1}^{+\infty} a_{nw}^{v}\int_0^{+\infty} L_w(t)\frac{dt}{t} - 1\right|\left|f(s_i) - f(s_{i-1})\right|\right]^{p_+^2/p_-^2 p(x_{i-1})} \\ &\leq \frac{1}{3}\left|\sum_{w=1}^{+\infty} a_{nw}^{v}\int_0^{+\infty} L_w(t)\frac{dt}{t} - 1\right|\sum_{i=1}^{m,}\left[3\lambda\left|f(s_i) - f(s_{i-1})\right|\right]^{p_+^2/p_-^2 p(x_{i-1})} \\ &\leq \frac{\varepsilon}{3}V^{p_+^2/p_-^2 p(x_{i-1})}[\lambda f] \\ &\leq \frac{\varepsilon}{3}V^{p(\cdot)}[\overline{\lambda}f] \end{aligned}$$

where $V^{p(\cdot)}\left[\overline{\lambda}f\right] < +\infty$ for some $\overline{\lambda} > 0$. The proof is completed. □

4 Rate of approximation

In this section, we will investigate the order of approximation of the operators by using suitable Lipschitz class of functions of $AC^{p(\cdot)}(\mathbb{R}_+)$. For $0 < \alpha \leq 1$ we denote the Lipschitz-type class

$$\begin{aligned} &V^{p(\cdot)}Lip(\alpha) \\ &= \left\{f \in AC^{p(\cdot)}(\mathbb{R}_+) : \exists\ \rho > 0 \text{ such that} V^{p(\cdot)}\left[\rho\left(\tau_t f - f\right)\right] = O\left(\left|logt\right|^{\alpha}\right) \text{ as } |1-t| \to 0\right\}, \end{aligned}$$

is local on bounded intervals and for $0 < \alpha \leq 1$ we will assume that

$$\sum_{w=1}^{+\infty} a_{nw}^{v}\int_0^{+\infty} L_w(t)\frac{dt}{t} - 1 = O(n^{-\alpha}) \text{ as } n \to +\infty \left(\text{uniformly in } v\right), \tag{4.1}$$

for any $\delta > 0$,

$$\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|<\delta} |L_w(t)|\left|logt\right|^{\alpha}\frac{dt}{t} = O(n^{-\alpha}) \text{ as } n \to +\infty \left(\text{uniformly in } v\right), \tag{4.2}$$

$$\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|\geq\delta} |L_w(t)|\frac{dt}{t} = O(n^{-\alpha}) \text{ as } n \to +\infty \left(\text{uniformly in } v\right), \tag{4.3}$$

Theorem 4.1 *Let* $\mathcal{A} = \{[a_{nw}^{\nu}]\}$ *be a nonnegative regular summability method. Assume that* (4.1), (4.2), (4.3) *holds. Let* $\{\delta_w\}$ *be a positive null sequence satisfying the property*

$$\frac{V^{p(\cdot)}[\lambda G_w]}{\gamma m(J)} \leq \delta_w \tag{4.4}$$

for every $w \in \mathbb{N}$ *and for every bounded interval* $J \subset \mathbb{R}$, *and*

$$\sum_{w=1}^{\infty} a_{nw}^{\nu}\delta_w = O(n^{-\alpha}) \quad \text{as } w \to +\infty (\text{uniformly in } \nu). \tag{4.5}$$

Then for every $f \in V^{p(\cdot)}Lip(\alpha)$,

$$V^{p(\cdot)}[\lambda(\mathcal{T}_{n,\nu}(f) - f)] = O(n^{-\alpha})$$

as $n \to +\infty$ *uniformly in* ν *holds for* $\lambda > 0$.

Proof We know from the Theorem 3.4 that

$$\begin{aligned} V^{p(\cdot)}[\lambda(\mathcal{T}_{n,\nu}(f) - f)] &\leq \frac{1}{3M}\sum_{w=1}^{+\infty} a_{nw}^{\nu} V^{p_+^2/p_-^2 p(x_{i-1})}[3MD\lambda(H_w(f) - f)] \\ &+ \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_0^{+\infty} |L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f - f\right)\right]\frac{dt}{t} \\ &+ \frac{1}{3}\left|\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_0^{+\infty} L_w(t)\frac{dt}{t} - 1\right| V^{p_+^2/p_-^2 p(x_{i-1})}[3\lambda f] = I_1 + I_2 + I_3. \end{aligned}$$

Now from (4.4) we obtain that $V^{p(\cdot)}[\lambda(H_w f - f)] \leq \delta_w$, and (see [15]), for every $\gamma > 0$, there exists a $\lambda > 0$, such that

$$V^{p_+^2/p_-^2 p(x_{i-1})}[3MD\lambda(H_w(f) - f)] \leq V^{p(\cdot)}[\gamma f]\delta_w$$

where $V^{p(\cdot)}[\gamma f] < +\infty$, $\lambda \leq \frac{\overline{\lambda}}{3MD}$. Then from (4.5) it follows that

$$I_1 = O(n^{-\alpha}) \text{ as } n \to +\infty \left(\text{uniformly in } \nu\right).$$

Since $f \in V^{p(\cdot)}Lip(\alpha)$, then for some $\rho > 0$, there exists δ, $B > 0$ such that

$$V^{p(\cdot)}\left[\rho(\tau_t f - f)\right] \leq B\left|logt\right|^{\alpha}$$

whenever $|1 - t| < \delta$. Therefore

$$\begin{aligned} I_2 &\leq \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_{|1-t|<\delta} |L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f - f\right)\right]\frac{dt}{t} \\ &+ \frac{1}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{\nu}\int_{|1-t|\geq\delta} |L_w(t)|\, V^{p_+^2/p_-^2 p(x_{i-1})}\left[3MD\lambda\left(\tau_t f - f\right)\right]\frac{dt}{t} \end{aligned}$$

$$\leq \frac{B}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|<\delta}|L_w(t)|\left|logt\right|^{\alpha}\frac{dt}{t}+\frac{V^{p(\cdot)}[3MD\lambda f]}{3MD}\sum_{w=1}^{+\infty} a_{nw}^{v}\int_{|1-t|\geq\delta}|L_w(t)|\frac{dt}{t}.$$

Using (4.2), (4.3) and (3.2), we get

$$I_2 = O(n^{-\alpha}) \text{ as } n\to+\infty \left(\text{uniformly in } v\right).$$

Finally, from (4.1), we conclude that

$$I_3 = O(n^{-\alpha}) \text{ as } n\to+\infty \left(\text{uniformly in } v\right). \qquad \square$$

Corollary 4.2 *Consider the Cesàro matrix summability* $\mathcal{A}=\{C_1\}=\{[c_{nw}]\}$, *given by* [39]

$$c_{nw}=\begin{cases}\frac{1}{n}, \text{ if } 1\leq w\leq n,\\ 0, \text{ otherwise,}\end{cases}$$

which gives the arithmetic mean convergence. Then we have the following approximations for $f\in AC^{p(\cdot)}(\mathbb{R}_+)$

$$\lim_{n\to+\infty} V^{p(\cdot)}\left[\lambda\left(\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right)\right]=0$$

and for every $f\in V^{p(\cdot)}Lip(\alpha)$,

$$V^{p(\cdot)}\left[\lambda\left(\frac{T_1(f)+T_2(f)+\cdots+T_n(f)}{n}-f\right)\right]=O(n^{-\alpha})$$

as $n\to+\infty$, *where* $T_n(f)$ *is of the form* (2.1).

5 Conclusion

The work explored the approximation behaviour of Mellin convolution-type integral operators within the context of variable bounded variation spaces, emphasizing the role of Bell-type matrix summability methods. The results demonstrate that such summability techniques not only improve convergence but also offer better control over approximation accuracy, particularly when applied to functions from Lipschitz-type classes. By comparing the impact of both Bell-type and classical summability methods, it highlighted the refined influence of summability on operator performance. The findings are expected to contribute to a better understanding of approximation processes in non-uniform function spaces and help for broader applications in areas where function smoothness varies or classical methods are not so effective.

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Author contributions
H.D. introduced the research topic to M.P.D. and guided the preparation of the initial manuscript draft. M.P.D. subsequently conducted the formal analysis and developed the proofs under the continuous supervision of H.D. Both authors reviewed the manuscript and approved it for submission.

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The authors declare no competing interests.

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APPROXIMATION OF CONVOLUTION-TYPE INTEGRAL OPERATORS ON VARIABLE BOUNDED VARIATION SPACES OF HIGHER ORDER USING SUMMABILITY METHODS

Manash Pratim Dutta and Hemen Dutta

Abstract. We investigate a new class of function spaces called variable bounded variation spaces of higher order by generalizing the classical notion of bounded variation. Our approach incorporates concepts from higher-order variations, bounded variation in the sense of generalized norms, and variations defined through modular function spaces. We focus on analysing the approximation properties of convolution-type nonlinear integral operators within these newly defined function spaces. In particular, we study these operators on variable bounded variation spaces of a fixed order in the sense of higher variations. Several approximation results are obtained using regular summability methods, highlighting the effectiveness of the proposed approach in improving convergence behaviour.

1. Introduction

Integral operators of convolution types are used not only in the classical and constructive approximation theory [**2**,**5**,**6**,**9**,**11**], but also in applied studies, such as image processing, signal processing, ultrasound diagnostics etc. [**12**, **22**, **25**]. Some of them have linear structure [**13**, **16**, **28**] and others are nonlinear [**1**, **3**]. To increase the speed of convergence of sequences, certain summability methods such as subsequence matrix transforms are applied in the context of approximation theory [**7**, **8**, **10**, **19**]. This is one of the primary reasons for applying summability methods in approximation theory. For better approximation results, we use the Bell type summability methods [**14**].

Assume that $\mathcal{A} = \{A^v\} = \{[a^v_{nk}]\} : (n, k, v \in \mathbb{N})$ be a family of infinite matrices of real or complex numbers. Let $x = \{x_k\}$ be a sequence, the $\mathcal{A}$-transform of x is a sequence $(\mathcal{A}x)^v_n$ which is defined by $(\mathcal{A}x)^v_n = \sum_{k=1}^{\infty} a^v_{nk} x_k$ $(n, v \in \mathbb{N})$ provided that

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the series converges for every n, v. Then x is called $\mathcal{A}$-summable to a number L whenever $\lim_{n\to\infty}(\mathcal{A}x)_n^v = L$ uniformly in $v \in \mathbb{N}$. Then we write $\mathcal{A} - \lim x = L$. A summability method $\mathcal{A} = \{A^v\} = \{[a_{nk}^v]\}$ is said to be regular if $\lim x_k = L$ implies $\mathcal{A} - \lim x = L$. We know that a method $\mathcal{A} = \{[a_{nk}^v]\}$ is regular if and only if the following conditions are satisfied:

(a) for each $n,\ v \in \mathbb{N}$, $\sum_{k=1}^{\infty} |a_{nk}^v| < \infty$, and there exist integer N, M such that $\sup_{n\geqslant N,\, v\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{nk}^v| \leqslant M$,
(b) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}^v = 1$ uniformly in v,
(c) for every $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{nk}^v = 0$ uniformly in v [**15**].

The concept of variation of a given function was initially introduced by Jordan in [**20**]. He proposed the concept of a function with bounded variation in 1880. Subsequently, Wiener [**26**], and Love and Young [**21**], Musielak and Orlicz [**23**], and Tonelli [**24**] expanded the Jordan variation in other directions. Currently, function spaces with variable exponents are a very active area of research interest. In particular, Castillo et al. introduced the variable bounded variation spaces in the sense of Wiener in [**17**]. A generalization of the spaces of bounded p-variation is found in [**26**], which generalized the classical BV space in the sense of Jordan. The analysis of variable bounded variation spaces is progressing rapidly due to its fundamental significance as well as crucial applications in a variety of domains [**18**, **27**].

Here we introduce the variable bounded variation spaces of order $m > 1$ by generalizing the bounded variation spaces. We have taken care of the Wiener's [**26**] notion of higher variations, p-bounded variation by Love and Young [**21**] and Musielak–Orlicz φ-variations [**23**].

2. Preliminaries and Definitions

Definition 2.1. By an admissible function $p(\cdot)$, we mean a function $p : \mathbb{R} \to [1, +\infty)$, such that $p_+ = \sup_{x\in\mathbb{R}} p(x)$ is finite. We use the notation $p_- = \inf_{x\in\mathbb{R}} p(x)$; obviously $p_- \geqslant 1$.

Definition 2.2. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $m \geqslant 1$, we define the $p(\cdot)$-variation of f on $\mathbb{R}$ in the Wiener's sense as follows

$$V_{\varphi,\Delta^m}^{p(\cdot)}[f] = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} [\varphi|\Delta^m f(t_i)|]^{p(x_{i-1})} = \sup_{\Pi_\delta^*} \sum_{i=1}^{n} \left[\varphi\left|\sum_{r=0}^{m} (-1)^r \binom{m}{r} f(t_{i-r})\right|\right]^{p(x_{i-1})},$$

where Π_δ^* is a tagged sequence, i.e., an increasing sequence $t_0 < t_1 < \cdots < t_n$, together with a finite sequence of numbers $x_0, x_1, \ldots, x_{n-1}$ subject to the condition $t_i \leqslant x_i \leqslant t_{i+1}$, for all $i = 0, \ldots, n-1$.

We consider the class of all convex functions $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that φ is a continuous, non-decreasing, $\varphi(0) = 0$, $\varphi(t) > 0$ for $t > 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

Definition 2.3. We define the space of functions of variable bounded variation of order m on $\mathbb{R}$ as

$$BV_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R}) = \left\{f\colon \mathbb{R} \to \mathbb{R} \mid V_{\varphi,\Delta^m}^{p(\cdot)}[\lambda f] < \infty,\ \text{for some } \lambda > 0\right\}.$$

This space endowed with the norm

$$\|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} = \inf\big\{\lambda > 0|\ V^{p(\cdot)}_{\varphi,\Delta^m}[f/\lambda, \mathbb{R}]\big\},$$

which is the Luxemburg type modular norm. Therefore,

$$BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) = \big\{f\colon \mathbb{R} \to \mathbb{R} \mid \|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} < +\infty\big\}.$$

The above variable bounded variation space of order m is a generalization of the bounded variation space. If $p(\cdot) = 1$, $\varphi = 1$, $m = 1$, the variable bounded variation of order m reduces to the Jordan variation [**20**], while this space coincides with the Wiener p-variation [**26**] for $p(\cdot) = p$, $p > 1$, $\varphi = 1$, $m = 1$ and finally, if $p(\cdot) = 1$, $m = 1$, the variable bounded variation of order m reduces to the Musielak–Orlicz φ-variation [**7**].

Definition 2.4. By modulus of $p(\cdot)$-continuity of the function $f\colon \mathbb{R} \to \mathbb{R}$ we mean

$$\omega^{p(\cdot)}_{\delta,\Delta^m}(f) = \sup_{\Pi^*_\delta} \sum_{i=1}^{n} |\Delta^m f(t_i)|^{p(x_{i-1})},$$

where Π^*_δ is a tagged partition of $\mathbb{R}$ with mesh not greater than δ.

Definition 2.5. The space of φ-absolutely $p(\cdot)$-continuous function of order m is denoted by $AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and defined as

$$AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) = \Big\{f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \text{ and } \lim_{\delta\to 0} \omega^{p(\cdot)}_{\delta,\Delta^m}(\lambda f) = 0\Big\},$$

for some $\lambda > 0$, where Π^*_δ is a tagged partition of $\mathbb{R}$ with mesh, at most δ.

It is not hard to see that, the above definition reduces to the φ-absolute continuity, if $m = 1$, $\lambda = 1$ and $p(\cdot) = 1$ and it is denoted by $AC_\varphi(\mathbb{R})$, the space of φ-absolutely continuous functions on $\mathbb{R}$. We observe that $AC_\varphi(\mathbb{R}) \subset AC^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.

We shall study the approximation properties in $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ of the classical nonlinear convolution integral operators defined in [**7**] (in the case of $\mathcal{A} = \{I\}$, the identity matrix, then we get the operators which were considered in [**1**, **3**]).

$$\mathcal{T}_{n,v}(f;s) = \sum_{k=1}^{\infty} a^v_{nk} T_k(f;s), \quad \text{where } T_k(f;s) = \int_{\mathbb{R}} K_k(t, f(s-t))dt.$$

i.e.,

$$\mathcal{T}_{n,v}(f;s) = \sum_{k=1}^{\infty} a^v_{nk} \int_{\mathbb{R}} K_k(t, f(s-t))dt,\ n, v \in \mathbb{N}.$$

Here the kernel $K_k\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a family of measurable functions such that $K_k(t,s) = L_k(t)H_k(s)$ for every $t, s \in \mathbb{R}$, where $L_k \in L^1(\mathbb{R})$ and $\{H_k\}$ is a family of functions such that $H_k(0) = 0$ having the Lipschitz property,

$$|H_k(x) - H_k(y)| \leqslant C|x - y|$$

for some $C > 0$ and for every $k \in \mathbb{N}$.

To avoid repetition, we present the following conditions here for use in subsequent parts of the paper:

(i) There exists a constant $D > 0$, such that $\|L_k\|_1 \leqslant D < \infty$, $\forall\, k \in \mathbb{N}$.

(ii) $\mathcal{A} - \lim \left(\int_{\mathbb{R}} L_k(t)dt\right) = 1$,

(iii) for any fixed $\delta > 0$, $\mathcal{A} - \lim \left(\int_{|t|\geqslant\delta} |L_k(t)|\, dt\right) = 0$.

(iv) Let $G_k(u) = H_k(u) - u,\ u \in \mathbb{R}$, for every $\gamma > 0$, there exists a $\lambda > 0$, such that $\lim_{k\to\infty} \frac{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda G_k]}{\varphi(\gamma m(J))} = 0$ uniformly with respect to every bounded interval $J \subseteq \mathbb{R}$, which means that for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$, $\frac{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda G_k]}{\varphi(\gamma m(J))} < \epsilon$, where $m(J)$ is the length of the interval J.

3. Convergence Results

PROPOSITION 3.1. *Let p and q be admissible functions. If $p(\cdot) \leqslant q(\cdot)$, then $V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] \geqslant V^{q(\cdot)}_{\varphi,\Delta^m}[\lambda f]$ for some $\lambda > 0$ and therefore $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \subset BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.*

PROOF. Let $f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$, then by the definition, for some $\lambda > 0$

$$V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] = \sup_{\Pi^*_\delta} \sum_{i=1}^{n} \left[\varphi|\Delta^m(\lambda f)(t_i)|\right]^{p(x_{i-1})} < \infty$$

where Π^*_δ is a tagged sequence $t_0 < t_1 < \cdots < t_n$, $x_0 < \cdots < x_{n-1}$ on $\mathbb{R}$. We take

$$\|f\|_{BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})} = \inf\left\{\lambda > 0 \mid V^{p(\cdot)}_{\varphi,\Delta^m}[f/\lambda] \leqslant 1\right\} = 1.$$

Since $p(x_{i-1}) \leqslant q(x_{i-1})$ and the function $t \to t^r$ is increasing for $t \geqslant 0$ and $r > 0$, we obtain

$$\sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{q(x_{i-1})} \leqslant \sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{p(x_{i-1})}.$$

Now, taking supremum on both sides, we get

$$\sup_{\Pi^*_\delta} \sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{q(x_{i-1})} \leqslant \sup_{\Pi^*_\delta} \sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(t_i)|)]^{p(x_{i-1})}$$

$$V^{q(\cdot)}_{\varphi,\Delta^m}[\lambda f] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f] < \infty.$$

Hence, $f \in BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and therefore $BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}) \subset BV^{q(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$. □

PROPOSITION 3.2. *If $f_1, \ldots, f_r \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R}),\ r \in \mathbb{N}$, then $f_1 + \cdots + f_r \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ and for some $\lambda > 0$ we have*

$$V^{p(\cdot)}_{\varphi,\Delta^m}\left[\lambda\left(\sum_{j=1}^{r} f_j\right)\right] \leqslant r^{p_+-1} \sum_{j=1}^{r} V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j].$$

PROOF. Let $t_0 < t_1 < t_2 < \cdots < t_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence. Let $\lambda > 0$ be such that $V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j] < \infty$ for every $j = 1, \ldots, r$. By the monotonicity and convexity of the power function $u^{p(\cdot)}$, $u \geqslant 0$ we have

$$\begin{aligned}\sum_{i=1}^{n}\left[\varphi\left|\Delta^m\lambda\sum_{j=1}^{r}f_j(t_i)\right|\right]^{p(x_{i-1})} &\leqslant \sum_{i=1}^{n}\left[\varphi\sum_{j=1}^{r}|\Delta^m\lambda f_j(t_i)|\right]^{p(x_{i-1})}\\ &\leqslant \sum_{i=1}^{n}\left\{\frac{1}{r}[r\lambda\varphi|\Delta^m f_1(t_i)|]^{p(x_{i-1})} + \cdots + \frac{1}{r}[r\lambda\varphi|\Delta^m f_r(t_i)|]^{p(x_{i-1})}\right\}\\ &\leqslant r^{p(x_{i-1})-1}\sum_{i=1}^{n}\{[\lambda\varphi|\Delta^m f_1(t_i)|]^{p(x_{i-1})} + \cdots + [\lambda\varphi|\Delta^m f_r(t_i)|]^{p(x_{i-1})}\}.\end{aligned}$$

Since $p(x_{i-1}) \leqslant p_+$ and passing supremum over all the tagged sequences in $\mathbb{R}$

$$\begin{aligned}V^{p(\cdot)}_{\varphi,\Delta^m}\left[\lambda\left(\sum_{j=1}^{r}f_j\right)\right] &\leqslant r^{p_+-1}\left\{V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_1(t_i)] + \cdots + V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_r(t_i)]\right\}\\ &\leqslant r^{p_+-1}\sum_{j=1}^{r}V^{p(\cdot)}_{\varphi,\Delta^m}[\lambda f_j].\end{aligned}$$

□

In order to operate using the convolution integral operators, we establish a relationship between $V^{p(\cdot)}_{\varphi,\Delta^m}[f]$ and its shifted form $\tau_t f(u) = f(u-t)$, $t, u \in \mathbb{R}$.

PROPOSITION 3.3. *For every $t \in \mathbb{R}$, $\tau_t f \in BV^{p(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$ if and only if $f \in BV^{\tau-tp(\cdot)}_{\varphi,\Delta^m}(\mathbb{R})$.*

PROOF. Let $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ be a tagged sequence in $\mathbb{R}$; then for $t \in \mathbb{R}$, $s_0 - t < s_1 - t < \cdots < s_n - t$, $x_0 - t < \cdots < x_{n-1} - t$ is again a tagged sequence. Therefore

$$\sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i - t)|)]^{p(x_{i-1})} = \sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i - t)|)]^{p((x_{i-1}-t)+t)} \leqslant V^{p(\cdot+t)}_{\varphi,\Delta^m}[f]$$

and passing to the supremum over all the tagged sequences in $\mathbb{R}$

$$V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f] \leqslant V^{\tau-tp(\cdot)}_{\varphi,\Delta^m}[f].$$

Again, if $s_0 < s_1 < \cdots < s_n$, $x_0 < x_1 < \cdots < x_{n-1}$ is a tagged sequence in $\mathbb{R}$, then for $t \in \mathbb{R}$, $s_0 + t < s_1 + t < \cdots < s_n + t$, $x_0 + t < \cdots < x_{n-1} + t$, is also a tagged sequence. Then,

$$\sum_{i=1}^{n}[\varphi(|\Delta^m f(s_i)|]^{p(x_{i-1}+t)} = \sum_{i=1}^{n}[\varphi(|\Delta^m f((s_i + t) - t)]^{p(x_{i-1}+t)} \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f].$$

Now, passing the supremum over all the tagged sequences, we have

$$V^{\tau-tp(\cdot)}_{\varphi,\Delta^m}[f] \leqslant V^{p(\cdot)}_{\varphi,\Delta^m}[\tau_t f].$$

□

THEOREM 3.4. *Let $\mathcal{A} = \{[a_{nk}^v]\}$ be a non-negative regular summability method and condition* (i) *holds. Then for $f \in BV_{\varphi,\Delta^m}^{p(\cdot)}$ there exists a $\mu > 0$ such that*

$$V_{\varphi,\Delta^m}^{p_+/p_-p(\cdot)}[\mu \mathfrak{T}_{n,v}(f)] \leqslant V_{\varphi,\Delta^m}^{p(\cdot)}[\lambda f];$$

$\mathfrak{T}_{n,v}$ maps from $BV_{\varphi,\Delta^m}^{p(\cdot)}$ into $BV_{\varphi,\Delta^m}^{p_+/p_-p(\cdot)}$, and $\lambda > 0$ is such that $V_{\varphi,\Delta^m}^{p(\cdot)}[\lambda f] < +\infty$.

PROOF. Let $s_0 < s_1 < \cdots < s_n, , x_0 < \cdots < x_{n-1}$ be a tagged sequence in $\mathbb{R}$. Then for $\mu > 0$, we have

$$\begin{aligned} U &= \sum_{i=1}^{n} \left[\varphi\left(|\Delta^m \mu \mathfrak{T}_{n,v} f(s_i)|\right)\right]^{p_+/p_-p(x_{i-1})} \\ &= \sum_{i=1}^{n} \left[\varphi\left(\mu \left| \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} L_k(t)(\Delta^m H_k(f(s_i - t)))dt \right|\right)\right]^{p_+/p_-p(x_{i-1})} \end{aligned}$$

Now, using Jensen's inequality and convexity property of the power function $u^{p_+/p_-p(\cdot)}$, $u \geqslant 0$, and condition (i), we get

$$\begin{aligned} &\leqslant \frac{1}{D} \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| \sum_{i=1}^{n} [\varphi \mu |\Delta^m H_k(f(s_i - t))|]^{p_+/p_-p(x_{i-1})} dt \\ &\qquad \leqslant \frac{1}{D} \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| V_{\varphi,\Delta^m}^{p_+/p_-p(x_{i-1})}[\mu DC f(s_i - t)]\, dt \\ &\qquad \leqslant \frac{1}{D} \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| V_{\varphi,\Delta^m}^{p_+/p_-p(x_{i-1})}[\mu DC \tau_t f]\, dt \end{aligned}$$

where $\tau_t f(u) = f(u - t)$, $t, u \in \mathbb{R}$ and by Proposition 3.3

$$U \leqslant \frac{1}{D} \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| V_{\varphi,\Delta^m}^{p_+/p_-p(\cdot+t)}[\mu DC f]\, dt.$$

Since for every $t \in \mathbb{R}$, $p_+/p_-p(\cdot + t) \geqslant p(\cdot)/p_-p(\cdot + t) \geqslant p(\cdot)$, then by Proposition 3.1 and condition (i) we have

$$U \leqslant \frac{1}{D} \sum_{k=1}^{\infty} a_{nk}^v \int_{\mathbb{R}} |L_k(t)| V_{\varphi,\Delta^m}^{p(\cdot)}[\mu DC f]\, dt \leqslant V_{\varphi,\Delta^m}^{p(\cdot)}[\mu DC f].$$

Therefore, $0 < \mu < \frac{\lambda}{DC}$, passing to the supremum over all the possible tagged sequences in $\mathbb{R}$, we conclude that $V_{\varphi,\Delta^m}^{p_+/p_-p(\cdot)}[\mu \mathfrak{T}_{n,v}(f)] \leqslant V_{\varphi,\Delta^m}^{p(\cdot)}[\lambda f]$. □

THEOREM 3.5. *If $f \in AC_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R})$, then for some $\lambda > 0$, there holds*

$$\lim_{\delta \to 0} \omega_{\delta,\Delta^m}^{p(\cdot)p_+^2/p_-^2}(\lambda f, \delta) = 0$$

or equivalently,

$$\lim_{t \to 0} V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot - t) - f)] = 0.$$

PROOF. Let $\lambda > 0$ be fixed. For a fixed tagged sequence $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$

$$V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot-t)-f)(s_i)] = \sup_{\Pi_\delta^*}\sum_{i=1}^{n}[\varphi(|\Delta^m(\lambda f)(s_i-t)-\Delta^m(\lambda f)(s_i)|)]^{p(\cdot)p_+^2/p_-^2}$$

Since, Δ^m is a linear operator, so

$$\begin{aligned}\Delta^m[f(\cdot-t)-f](s_i) &= \Delta^m f(s_i-t)-\Delta^m f(s_i),\\ |\Delta^m[f(\cdot-t)-f](s_i)| &\leqslant |\Delta^m f(s_i-t)-\Delta^m f(s_i)|,\\ V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot-t)-f)(s_i)] &\leqslant \sup_{\Pi_\delta^*}\sum_{i=1}^{n}[\varphi(\lambda|\Delta^m f(s_i-t)-\Delta^m f(s_i)|)]^{p(\cdot)p_+^2/p_-^2}.\end{aligned}$$

Since $f \in AC_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R})$, it follows that $\Delta^m f$ uniformly continuous on $\mathbb{R}$, hence as $t \to 0$ we have $|\Delta^m f(s_i-t)-\Delta^m f(s_i)| \to 0$. Moreover, φ is continuous, increasing, $\varphi(0) = 0$ and for all $x \in \mathbb{R}$, $p(\cdot)p_+^2/p_-^2 \geqslant p(\cdot) \geqslant 1$, each term in the summation tends to zero as $t \to 0$. Thus, we have $\lim_{t\to 0} V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(f(\cdot-t)-f)] = 0$. □

THEOREM 3.6. *Let $\mathcal{A} = \{[a_{nk}^v]\}$ be a nonnegative regular summability method. Assume that conditions* (i)–(iv) *hold. If $f \in AC_{\varphi,\Delta^m}^{p(\cdot)}(\mathbb{R})$, then there exists $\lambda > 0$, such that $\lim_{n\to\infty} V_{\varphi,\Delta^m}^{p(\cdot)p_+^2/p_-^2}[\lambda(\mathcal{T}_{n,v}(f)-f)] = 0$ uniformly in $v \in \mathbb{N}$.*

PROOF. For a fixed tagged sequence $s_0 < s_1 < \cdots < s_n$, $x_0 < \cdots < x_{n-1}$ and $\lambda > 0$, there holds

$$\begin{aligned}U &= \sum_{i=1}^{n}[\varphi(\lambda|\Delta^m(\mathcal{T}_{n,v}f-f)(s_i)|)]^{p(\cdot)p_+^2/p_-^2}\\ &= \sum_{i=1}^{n}\Bigg[\varphi\Bigg|\Delta^m\lambda\Bigg(\sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}L_k(t)\big(H_k(f(s_i-t))-f(s_i-t)\big)\,dt\\ &\qquad + \sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}L_k(t)\big(f(s_i-t)-f(s_i)\big)\,dt\\ &\qquad + \Bigg(\sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}L_k(t)\,dt-1\Bigg)f(s_i)\Bigg)\Bigg|\Bigg]^{p(\cdot)p_+^2/p_-^2}\\ &\leqslant \frac{1}{3}\sum_{i=1}^{n}\Bigg[\varphi\Bigg(3\lambda\sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}|L_k(t)|\Big|\Delta^m\big(H_k(f(s_i-t))-f(s_i-t)\big)\Big|\,dt\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2}\\ &\quad + \frac{1}{3}\sum_{i=1}^{n}\Bigg[\varphi\Bigg(3\lambda\sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}|L_k(t)|\Big|\Delta^m\big(f(s_i-t)-f(s_i)\big)\Big|\,dt\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2}\\ &\quad + \frac{1}{3}\sum_{i=1}^{n}\Bigg[\varphi\Bigg(3\lambda\Bigg|\sum_{k=1}^{\infty}a_{nk}^v\int_{\mathbb{R}}L_k(t)dt-1\Bigg||\Delta^m f(s_i)|\Bigg)\Bigg]^{p(\cdot)p_+^2/p_-^2} = U_1+U_2+U_3.\end{aligned}$$

Let $p(\cdot)p_+^2/p_-^2 = \alpha$; using Jensen's inequality in U_1 with (i) and (b), there holds

$$\begin{aligned} U_1 &\leqslant \frac{1}{3M}\sum_{i=1}^{n}\left[\varphi\left(3M\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|\big|\Delta^m\big(H_k\big(f(s_i-t)\big)-f(s_i-t)\big)\big|\right)\right]^{\alpha}dt \\ &\leqslant \frac{1}{3MD}\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|\sum_{i=1}^{n}\left[\varphi\big(3MD\lambda\big|\Delta^m\big(H_k\big(f(s_i-t)\big)-f(s_i-t)\big)\big|\big)\right]^{\alpha}dt \\ &\leqslant \frac{1}{3MD}\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)|V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\big(f(\cdot-t)\big)-f(\cdot-t))]dt. \end{aligned}$$

Considering that

$$V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\left(f(\cdot-t)\right)-f(\cdot-t))] = V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k\left(f\right)-f)]$$

for every $t\in\mathbb{R}$, we get

$$U_1 \leqslant \frac{1}{3M}\sum_{k=1}^{\infty}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)].$$

Now, considering condition (iv) in Lemma 1 of [**4**], there exists a $k_0\in\mathbb{N}$ such that for all $k\geqslant k_0$, $V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] < \varepsilon V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]$ for some $\lambda>0$. Then, divide the sum given above into two parts as follows

$$\begin{aligned} U_1 &\leqslant \frac{1}{3M}\Bigg(\sum_{k=1}^{k_0-1}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] \\ &\qquad + \sum_{k=k_0}^{\infty}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)]\Bigg) \\ &\leqslant \frac{1}{3M}\Bigg(\sum_{k=1}^{k_0-1}a_{nk}^{v}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)] + \varepsilon V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]\sum_{k=k_0}^{\infty}a_{nk}^{v}\Bigg) \\ &= U_1^1 + U_1^2. \end{aligned}$$

Now, defined $P = \max_{k\in\{1,2,\ldots,k_0-1\}}V_{\varphi,\Delta^m}^{\alpha}[3MD\lambda(H_k(f)-f)]$, from (c) we get

$$U_1^1 \leqslant \frac{P}{3M}\sum_{k=1}^{k_0-1}a_{nk}^{v} < \frac{P}{3M}(k_0-1)\varepsilon.$$

In U_1^2, from the regularity property (a), we get $U_1^2 \leqslant \frac{\epsilon}{3}V_{\varphi,\Delta^m}^{p(\cdot)}[\gamma f]$. For U_2, by the Jensen's inequality we observe that

$$\begin{aligned} U_2 &\leqslant \frac{1}{3M}\sum_{i=1}^{n}\left[\varphi\left(3M\lambda\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}|L_k(t)||\Delta^m(f(s_i-t)-f(s_i))|\right)\right]^{\alpha}dt \\ &\leqslant \frac{\lambda}{3M}\sum_{i=1}^{n}\left[\left(3M\sum_{k=1}^{\infty}a_{nk}^{v}\left\{\int_{|t|<\delta}+\int_{|t|\geqslant\delta}\right\}|L_k(t)||\Delta^m(f(s_i-t)-f(s_i))|\right)\right]^{\alpha}dt \end{aligned}$$

$$\leqslant \frac{1}{3MD}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\sum_{i=1}^{n}[\varphi(3MD\lambda|\Delta^m(f(s_i-t)-f(s_i))|)]^{\alpha}dt$$
$$+\frac{1}{3MD}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\sum_{i=1}^{n}[\varphi(3MD\lambda|\Delta^m(f(s_i-t)-f(s_i))|)]^{\alpha}dt$$
$$\leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i))]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt$$
$$+\frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i))]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt \leqslant U_2^1+U_2^2.$$

Consider (i) and Theorem 3.5, we get

$$U_2^1 \leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(f(s_i-t)-f(s_i)]\sum_{k=1}^{n} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt$$
$$\leqslant \frac{1}{3MD}\omega^{\alpha}_{\delta,\Delta^m}(3MD\lambda f,\delta)\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|<\delta}|L_k(t)|\,dt<\frac{\epsilon}{3},$$

and consider U_2^2, by Propositions 3.1 and 3.2

$$U_2^2 \leqslant \frac{1}{3MD}V^{\alpha}_{\varphi,\Delta^m}[3MD\lambda(\tau_t f-f)]\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt$$
$$\leqslant \frac{1}{3MD}2^{p_+-1}\left\{V^{\alpha}_{\varphi,\Delta^m}[6MD\lambda\tau_t f]+V^{\alpha}_{\varphi,\Delta^m}[6MD\lambda f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt$$
$$\leqslant \frac{1}{3MD}2^{p_+}\left\{V^{p(\cdot)p_+/p_-}_{\varphi,\Delta^m}[6MD\lambda f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt$$
$$\leqslant \frac{1}{3MD}2^{p_+}\left\{V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]\right\}\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt$$

for $0<\lambda<\frac{\mu}{6MD}$, where μ is such that $V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]<\infty$. By assumption (iii), there exists $k>0$, such that

$$\sum_{k=1}^{\infty} a_{nk}^{v}\int_{|t|\geqslant\delta}|L_k(t)|\,dt<\frac{3MD\epsilon}{2^{(p_++1)}V^{p(\cdot)}_{\varphi,\Delta^m}[\mu f]}.$$

Therefore, $U_2^2<\epsilon/2$.

Now, from (iii), for every $0<\epsilon<1$, there exists n_0, such that, for all $n\geqslant n_0$

$$\left|\sum_{k=1}^{n} a_{nk}^{v}\int_{\mathbb{R}}L_k(t)\,dt-1\right|<\epsilon<1.$$

We consider this in U_3, by the convexity of φ we observe that

$$U_3 \leqslant \frac{1}{3}\sum_{i=1}^{n}\left|\varphi\left(3\lambda\left[\sum_{k=1}^{n}a_{nk}^{v}\int_{\mathbb{R}}L_k(t)dt-1\right]\right)\Delta^m f(s_i)\right|^{\alpha}$$
$$\leqslant \frac{1}{3}\left|\sum_{k=1}^{\infty}a_{nk}^{v}\int_{\mathbb{R}}L_k(t)dt-1\right|\sum_{i=1}^{n}[\varphi(3\lambda|\Delta^m f(s_i)|)]^{\alpha} \leqslant \frac{\epsilon}{3}V_{\varphi,\Delta^m}^{p(\cdot)}[3\lambda f]$$

where $V_{\varphi,\Delta^m}^{p(\cdot)}[3\lambda f] < \infty$ for some $\lambda > 0$. Therefore, the proof is complete. □

COROLLARY 3.7. *For $m = 1$ and $p(\cdot) = 1$, the space of φ-absolutely $p(\cdot)$ continuous functions of order m becomes the space of φ-absolutely continuous functions and above Theorem* 3.6 *reduces to Theorem* 3.2 *in* [**7**]. *(i.e., if $f \in AC_\varphi(\mathbb{R})$ then there exists $\mu > 0$ such that $\lim_{n\to\infty} V_\varphi[\mu(\mathcal{T}_{n,v}f - f)] = 0$, uniformly in $v \in \mathbb{N}$.)*

4. Conclusion

We observe that classical bounded variation spaces are mapped into themselves by convolution type nonlinear integral operators with regular summability methods. They map $BV_{\varphi,\Delta^m}^{p(\cdot)}$ into $BV_{\varphi,\Delta^m}^{p_+/p_-p(\cdot)}$ however, in the generalised context of variable bounded variations of higher order, capturing more complex function behaviour while maintaining the modular framework. This pattern opens up new paths for the study of integral operators under more adaptable and realistic functional contexts, in addition to extending classical approximation theory. Adding the variable exponent $p(\cdot)$ and a higher order difference operator greatly enhances the theory and increases its applicability in a wide range of fields.

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