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A MODIFIED VISCOSITY TYPE INERTIAL SUBGRADIENT EXTRAGRADIENT ALGORITHM FOR NONMONOTONE EQUILIBRIUM PROBLEMS AND APPLICATION TO CARDIOVASCULAR DISEASE DETECTION

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Abstract. This paper proposes a new modified viscosity type inertial subgradient extragradient algorithm for solving nonmonotone equilibrium problems in real Hilbert spaces. It proves the strong convergence theorem of the algorithm under suitable conditions. In addition, we utilize the proposed algorithm to solve the cardiovascular disease classification in machine learning by using the cardiovascular disease dataset from the Kaggle website, which is cleaned by a smooth data technique in the MATLAB program. The comparisons show that our algorithm is more efficient than other machine learning methods.

1. **Introduction.** Throughout this paper, let $\mathbb{H}$ is a real Hilbert space and $\mathbb{C}$ is a nonempty closed and convex subset of $\mathbb{H}$. The equilibrium problem (EP) is to find an element $\vartheta^* \in \mathbb{C}$ introduced by Ky Fan [4] such that

$$g(\vartheta^*, \delta) \geq 0, \forall \delta \in \mathbb{C}, \tag{1}$$

where g satisfies $g(\vartheta, \vartheta) = 0$ for all $\vartheta \in \mathbb{H}$ and $g : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ is a bifunction. $EP(g, \mathbb{C})$ is defined for a solution set of EP(1). Equilibrium problems can be modeled to solve problems in applied mathematics, such as optimization problems, variational inequality, fixed point problems, vector and scalar minimization problems, and many more. The equilibrium problem for solving the cardiovascular disease dataset and other datasets is seen in [3, 15].

Many real world problems can be solved via the formulation in EP (1). Thus, many algorithms have been introduced to solve this problem (1). One of the famous algorithms introduced by Tran et al. [17] is the two-step extragradient method (TSEM). The TSEM was modified from the concept idea of the extragradient method (EM), which was presented by Korpelevich [7] for solving equilibrium problems such that the solution satisfies Lipschitz-type conditions and the bifunction is

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pseudomonotone. This method was generated by $\theta^0 \in \mathbb{H}$, and

$$\begin{cases} \gamma^k = argmin_{\gamma\in\mathbb{C}}\{\rho g(\theta^k,\gamma) + \frac{1}{2}\|\theta^k - \gamma\|^2\}, \\ \theta^{k+1} = argmin_{\gamma\in\mathbb{C}}\{\rho g(\gamma^k,\gamma) + \frac{1}{2}\|\theta^k - \gamma\|^2\}, \end{cases}$$

where $0 < \rho < \min\{\frac{1}{2c^1}, \frac{1}{2c^2}\}$, but they have the only weak convergence gained in Hilbert spaces under some condition on bifunction g and control step size parameters $\{b^k\}$. Moreover, using the TSEM, for each iteration it is necessary to calculate two projections on $\mathbb{C}$, which is hard when $\mathbb{C}$ is a complex structure.

The advantage of modifying the algorithm is obtaining an algorithm that converges strongly to the solution. One of the famous strongly convergent algorithms is the Halpern iteration [5]. Using the idea of this algorithm, Hieu [18] proposed the Halpern subgradient extragradient method (HSEM), which was generated by $\tau, \theta^1 \in \mathbb{H}$ and

$$\begin{cases} \gamma^k &= argmin_{\gamma\in\mathbb{C}}\left\{bg(\theta^k,\gamma) + \frac{1}{2}\|\theta^k - \gamma\|^2\right\}, \\ Q^k &= \{\delta \in \mathbb{H} : \langle(\theta^k - bq^k) - \gamma^k, \delta - \gamma^k\rangle\}, \quad q^k \in \partial_2 g(\theta^k,\gamma^k), \\ \nu^k &= argmin_{\gamma\in Q^k}\left\{bg(\gamma^k,\gamma) + \frac{1}{2}\|\theta^k - \gamma\|^2\right\}, \\ \theta^{k+1} &= d^k\tau + (1-d^k)\nu^k, \end{cases}$$

where $0 < b < \min\{\frac{1}{2c^1}, \frac{1}{2c^2}\}, 0 < d^k < 1, \lim_{k\to\infty} d^k = 0$, and $\sum_{k=1}^{\infty} d^k = \infty$.

Another famous strongly convergent algorithm is the viscosity method [8], in which Muangchoo [10] modified the HSEM to create the viscosity subgradient extragradient method (VSEM), which was generated by $\eta^1 \in \mathbb{H}$, and this method is defined by

$$\begin{cases} \delta^k &= argmin_{\delta\in\mathbb{C}}\{b^k g(\eta^k,\delta) + \frac{1}{2}\|\eta^k - \delta\|^2\} \\ \varepsilon^k &= \{t \in \mathbb{H} : \langle\eta^k - b^k\omega^k - \delta^k, t - \delta^k\rangle \le 0\}, \ \omega^k \in \partial_2 g(\eta^k,\delta^k) \\ t^k &= argmin_{\delta\in\varepsilon^k}\{eb^k g(\delta^k,\delta) + \frac{1}{2}\|\eta^k - \delta\|^2\} \\ \eta^{k+1} &= d^k G(\eta^k) + (1-d^k)t^k, \end{cases}$$

where $b^k > 0$, $0 < \sigma < \min\{1, \frac{1}{2c^1}, \frac{1}{2c^2}\}, e \in (0,\sigma)$, G is a strict contraction function such that the constant $\xi \in [0,1)$ and $\{d^k\}$ satisfies the same conditions of the HSEM. Moreover, the step size b^k was improved by updating for each iteration, and the Lipschitz constant of g was not required. The step size $\{b^k\}$ was defined by

$$b^{k+1} = \begin{cases} \min\{\sigma, \frac{eg(\delta^k,t^k)}{K^k}\}, & \text{if } \frac{eg(\delta^k,t^k)}{K^k} > 0, \\ b^0, & \text{otherwise,} \end{cases}$$

where $K^k = g(\eta^k,t^k) - g(\eta^k,\delta^k) - c^1\|\eta^k - \delta^k\|^2 - c^2\|t^k - \delta^k\|^2 + 1$.

One famous technique for speeding up the convergence of the algorithm, called the inertial technique, was introduced by Polyak [12] in the part of well-known heavy-ball method. Very recently, a subgradient extragradient methods with self-adaptive step size and inertial extrapolation step was modified by Shehu et al. [13]. This is to gain strong convergence to an element of $EP(g,\mathbb{C})$ when g is a pseudomonotone mapping, which satisfies the Lipchitz-type condition. This method

was defined by $\eta^0, \eta^1 \in \mathbb{H}$ and

$$\begin{cases} \alpha^k & = d^k\eta^0 + (1-d^k)\eta^k + s^k(\eta^k - \eta^{k-1}), \\ \delta^k & = argmin_{\delta\in\mathbb{C}}\left\{b^k g(\alpha^k,\delta) + \frac{1}{2}\|\alpha^k - \delta\|^2\right\}, \\ \varepsilon^k & = \{t \in \mathbb{H} : \langle(\alpha^k - b^k\omega^k) - \delta^k, t - \delta^k\rangle\}, \quad \omega^k \in \partial_2 g(\alpha^k,\delta^k), \\ t^k & = argmin_{\delta\in\varepsilon^k}\left\{b^k g(\delta^k,\delta) + \frac{1}{2}\|\alpha^k - \delta\|^2\right\}, \\ \eta^{k+1} & = (1-d)d^k + dt^k, \end{cases}$$

where the inertial parameter $s^k \in [0, \frac{1}{3})$, $d \in (0, \frac{1}{2}]$. The update step size $\{b^k\}$ was defined by

$$b^{k+1} = \begin{cases} \min\{\frac{e}{2}\frac{\|\alpha^k-\delta^k\|^2+\|t^k-\delta^k\|^2}{g(\alpha^k,t^k)-g(\alpha^k,\delta^k)-g(\delta^k,t^k)}, b^k\}, \\ \text{if } g(\alpha^k,t^k) - g(\alpha^k,\delta^k) - g(\delta^k,t^k) > 0, \\ b^k, \quad \text{otherwise}, \end{cases}$$

and $\{d^k\}$ still satisfies the same conditions of the HSEM.

In 2022, Suantai et al. [14] introduced the viscosity type inertial extragradient algorithm (VIE), where g is pseudomonotone and satisfies a Lipchitz-type condition. This method was defined by $\theta^0, \theta^1 \in \mathbb{H}$ and

$$\begin{cases} \alpha^k & = d^k G(\theta^k) + (1-d^k)\theta^k + n^k(\theta^k - \theta^{k-1}), \\ \gamma^k & = argmin_{\gamma\in\mathbb{C}}\{b^k g(\alpha^k,\gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\}, \\ \eta^k & = argmin_{\gamma\in\mathbb{C}}\{b^k g(\gamma^k,\gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\}, \\ \theta^{k+1} & = (1-p)\alpha^k + p\eta^k, \end{cases}$$

where $p \in (0, \frac{1}{2}], d^k \in (0,1), n^k \in [0, \frac{1}{3})$. With the same conditions on $\{d^k\}$ of the HSEM, the strong convergence was proved in Hilbert spaces.

In 2024, Yao et al. [20] proposed a proximal-type algorithm that uses one proximal point to solve the EP with pseudo-monotone bifunction, choosing $b^1 > 0, \eta^0, \tau^0 \in \mathbb{H}, and \theta^1 = \theta^0 \in \mathbb{C}$ as follows:

$$\begin{cases} \theta^k & = \tau^k + \delta(\eta^{k-1} - \tau^k), \\ \eta^k & = \theta^k + n^k(\theta^k - \theta^{k-1}), \\ \tau^{k+1} & = argmin_{\gamma\in\mathbb{C}}\{b^k g(\tau^k,\gamma) + \frac{1}{2}\|\gamma - \eta^k\|^2\}, \end{cases}$$

where $\sum_{k=1}^{\infty} t^k < \infty, \mu \in (0,1), \theta \in (0,1), -1 < n^k \leq n^{k+1} \leq 0$, and $\delta \in [\frac{(\sqrt{5}-1)}{2}, 1)$, with step sizes

$$b^{k+1} = \begin{cases} b^k + \tau^k, & \text{if } g(\tau^{k-1},\tau^{k+1}) - g(\tau^{k-1},\tau^k) - g(\tau^k,\tau^{k+1}) \leq 0, \\ \min\left\{\frac{\mu\theta}{4}\frac{(\|\tau^k-\tau^{k+1}\|^2+\|\tau^k-\tau^{k-1}\|^2)}{(g(\tau^{k-1},\tau^{k+1})-g(\tau^{k-1},\tau^k)-g(\tau^k,\tau^{k+1}))}, b^k + \tau^k\right\}, & \text{otherwise.} \end{cases}$$

In this work, we focus on modifying an algorithm to solve pseudomonotone equilibrium problems with applications in machine learning. We use an inertial technique with viscosity type subgradient extragradient methods to obtain a strong convergence theorem.

2. **Preliminaries.** A bifunction $g : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be:
(i) mototone on $\mathbb{C}$ if $g(\epsilon, \delta) \leq -g(\delta, \epsilon), \forall \epsilon, \delta \in \mathbb{C}$;
(ii) strongly monotone on $\mathbb{C}$ if there exists a constant $\varphi > 0$ such that

$$g(\epsilon, \delta) + g(\delta, \epsilon) \leq -\varphi\|\epsilon - \delta\|^2, \ \forall \epsilon, \delta \in \mathbb{C};$$

(iii) pseudomonotone on $\mathbb{C}$ if $g(\epsilon, \delta) \geq 0 \Longrightarrow g(\delta, \epsilon) \leq 0, \forall \epsilon, \delta \in \mathbb{C}$;
(iv) satisfying a Lipschitz-type condition on $\mathbb{C}$ if there exist two positive constants c^1, c^2 such that

$$c^1\|\epsilon - \delta\|^2 + c^2\|\delta - \eta\|^2 \geq g(\epsilon, \eta) - g(\epsilon, \delta) - g(\delta, \eta), \forall \epsilon, \delta, \eta \in \mathbb{C}.$$

The normal cone $N_{\mathbb{C}}$ to $\mathbb{C}$ at a point $\epsilon \in \mathbb{C}$ is defined by $N_{\mathbb{C}}(\epsilon) = \{m \in \mathbb{H} : \langle m, \epsilon - \delta\rangle \geq 0, \forall \delta \in \mathbb{C}\}$. For every $\epsilon \in \mathbb{H}$, the metric projection $P_{\mathbb{C}}\epsilon$ of ϵ onto $\mathbb{C}$ is the nearest point of ϵ in $\mathbb{C}$, that is, $P_{\mathbb{C}}\epsilon = argmin\{\|\delta - \epsilon\| : \delta \in \mathbb{C}\}$.
For each $\epsilon, \gamma \in \mathbb{H}$, the subdifferential of convex function $g(\gamma, .)$ at ϵ is defined by

$$\partial_2 g(\gamma, \epsilon) = \{a \in \mathbb{H} : g(\gamma, \delta) \geq \langle a, \delta - \epsilon\rangle + g(\gamma, \epsilon), \ \forall \delta \in \mathbb{H}\}.$$

In particular, for $\gamma \in \mathbb{C}$,

$$\partial_2 g(\gamma, \gamma) = \{a \in \mathbb{H} : g(\gamma, \delta) \geq \langle a, \delta - \gamma\rangle, \ \forall \delta \in \mathbb{H}\}.$$

We need the following lemmas to prove our main result.

Lemma 2.1. [1] *For each $\epsilon \in \mathbb{H}$ and $b > 0$,*

$$\frac{1}{b}\langle \epsilon - prox_{bg}(\epsilon), \delta - prox_{bg}(\epsilon)\rangle \leq g(\delta) - g(prox_{bg}(\epsilon)), \ \forall \delta \in \mathbb{C},$$

where $prox_{bg}(\epsilon) = argmin\{bg(\delta) + \frac{1}{2}\|\epsilon - \delta\|^2 : \delta \in \mathbb{C}\}$.

Lemma 2.2. [11] *Let $\{\phi^k\}$ and $\{\kappa^k\}$ be two sequences in $[0, +\infty)$ such that $\sum_{k=1}^{\infty} \phi^k < \infty$, and let $\{\lambda^k\}$ be a sequence of real numbers such that $0 \geq \limsup_{k\to\infty} \lambda^k$. If there is a positive integer k such that*

$$\kappa^{k+1} \leq (1 - q^k)\kappa^k + q^k\lambda^k + \phi^k,$$

where $\{q^k\}$ is a sequence in $(0, 1)$ such that $\sum_{k=1}^{\infty} q^k = \infty$, then $\lim_{k\to\infty} \kappa^k = 0$.

Lemma 2.3. [19] *Let $\{\psi^k\}$, $\{\lambda^k\}$, and $\{o^k\}$ be positive sequences, such that*

$$\psi^{k+1} \leq (1 + o^k)\psi^k + \lambda^k, \ k \geq 1.$$

If $\sum_{k=1}^{\infty} o^k < +\infty$ and $\sum_{k=1}^{\infty} \lambda^k < +\infty$, then $\lim_{k\to+\infty} \psi^k$ exists.

3. **Main results.** We modify the subgradient extragradient algorithm by the inertial technique and viscosity term in order to use it for solving equilibrium problems.

Modified viscosity type inertial subgradient extragradient algorithm (MVISE)

Initialization. Choose initials $\epsilon^0, \epsilon^1 \in \mathbb{H}$. Pick parameters $0 < b^k \leq b < \frac{1}{2\max\{c^1, c^2\}}$, $r \in (0, \frac{1}{2}]$, $n^k \in [0, \frac{1}{3})$, and $d^k \in (0, 1)$.
Step 1. Given the current iterates ϵ^{k-1} and $\epsilon^k (k \geq 1)$, compute

$$\alpha^k = d^k G(\epsilon^k) + (1 - d^k)\beta^k,$$

$$\gamma^k = argmin_{\gamma\in\mathbb{C}}\{b^k g(\alpha^k, \gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\},$$

where $\beta^k = \epsilon^k + n^k(\epsilon^k - \epsilon^{k-1})$, and γ^k is a solution if $\gamma^k = \alpha^k$.

Step 2. Choose $\delta^k \in \partial_2 g(\alpha^k, .)(\gamma^k)$ such that there exists $\zeta^k \in N_{\mathbb{C}}(\gamma^k)$ satisfying

$$\zeta^k = \alpha^k - b^k\delta^k - \gamma^k$$

and construct a half-space

$$\Gamma^k = \{\eta \in \mathbb{H} : \langle \alpha^k - b^k\delta^k - \gamma^k, \eta - \gamma^k\rangle \leq 0\}.$$

Compute

$$\eta^k = argmin_{\gamma\in\Gamma^k}\{b^k g(\gamma^k, \gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\}.$$

Step 3. Compute

$$\epsilon^{k+1} = (1 - r)\alpha^k + r\eta^k.$$

Set $k = k + 1$ and return to **Step 1.**

Setting $G(\epsilon) = \epsilon^0,\ \forall \epsilon \in \mathbb{H}$, we gain the following algorithm:
Modified Halpern inertial subgradient extragradient algorithm (MHISE)

Initialization. Choose initials $\epsilon^0, \epsilon^1 \in \mathbb{H}$. Pick parameters $0 < b^k \leq b < \frac{1}{2\max\{c^1,c^2\}}$, $r \in (0, \frac{1}{2}]$, $n^k \in [0, \frac{1}{3})$, and $d^k \in (0, 1)$.

Step 1. Given the current iterates ϵ^{k-1} and $\epsilon^k (k \geq 1)$, compute

$$\alpha^k = d^k\epsilon^0 + (1 - d^k)\beta^k,$$

$$\gamma^k = argmin_{\gamma\in\mathbb{C}}\{b^k g(\alpha^k, \gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\},$$

where $\beta^k = \epsilon^k + n^k(\epsilon^k - \epsilon^{k-1})$, and γ^k is a solution if $\gamma^k = \alpha^k$.

Step 2. Choose $\delta^k \in \partial_2 g(\alpha^k, .)(\gamma^k)$ such that there exists $\zeta^k \in N_{\mathbb{C}}(\gamma^k)$ satisfying

$$\zeta^k = \alpha^k - b^k\delta^k - \gamma^k$$

and construct a half-space

$$\Gamma^k = \{\eta \in \mathbb{H} : \langle \alpha^k - b^k\delta^k - \gamma^k, \eta - \gamma^k\rangle \leq 0\}.$$

Compute

$$\eta^k = argmin_{\gamma\in\Gamma^k}\{b^k g(\gamma^k, \gamma) + \frac{1}{2}\|\gamma - \alpha^k\|^2\}.$$

Step 3. Compute

$$\epsilon^{k+1} = (1 - r)\alpha^k + r\eta^k.$$

Set $k = k + 1$ and return to **Step 1.**

(A1) For $\epsilon \in \mathbb{H}$, $g(\epsilon, .)$ is convex, lower semicontinuous, and subdifferentiable on $\mathbb{H}$;

(A2) $g(., \delta)$ is sequentially weakly upper semicontinuous on $\mathbb{C}$ for each fixed point $\delta \in \mathbb{C}$, i.e.,
if $\{\epsilon^k\} \subset \mathbb{C}$ is a sequence converging weakly to $\epsilon \in \mathbb{C}$, then $g(\epsilon, \delta) \geq \limsup_{k\to\infty} g(\epsilon^k, \delta)$;

(A3) $G : \mathbb{H} \to \mathbb{H}$ is a contraction with contraction constant d;

(A4) g satisfies a Lipschitz-type condition on $\mathbb{H}$ with two constants c^1 and c^2;

(A5) $g(\epsilon, .)$ is continuous at some $\eta \in \mathbb{C}\ \forall \epsilon \in \mathbb{C}$ or g is pseudomonotone on $\mathbb{C}$ with $int(\mathbb{C}) \neq \emptyset$.

In this part, we assume the following condition below and we shall analyze the convergence of the Algorithm **MVISE**.

(i) $0 \leq n^k \leq n^{k+1} \leq n < \frac{1}{3}$ and $\lim_{k\to\infty} \frac{n^k}{d^k}\|\epsilon^k - \epsilon^{k-1}\| = 0$;

(ii) $EP(g, \mathbb{C}) \neq \emptyset$;

(iii) $\{d^k\} \subset (0, 1]$ is non-increasing with $\sum_{k=1}^{\infty} d^k = \infty$ and $\lim_{k\to\infty} d^k = 0$.

Next, we give some useful lemmas for proving our main results.

Lemma 3.1. *Suppose that Condition 3 and Condition 3 hold. Let $\{\epsilon^k\}$ be a sequence generated by Algorithm 3. Then, for all $\theta^* \in EP(g, \mathbb{C})$, there exists $\Lambda > 0$ such that*

$$\|\epsilon^{k+1} - \theta^*\|^2 \leq \|\alpha^k - \theta^*\|^2 - \|\epsilon^{k+1} - \alpha^k\|^2, \ k \geq \Lambda.$$

Proof. By Lemma 2.1, and the definition of γ^k, we get

$$\frac{1}{b^k}\langle \alpha^k - \gamma^k, \gamma - \gamma^k\rangle \leq g(\alpha^k, \gamma) - g(\alpha^k, \gamma^k), \ \forall \gamma \in \mathbb{C}. \tag{2}$$

Putting $\gamma = \eta^k$ in (2), we obtain

$$\frac{1}{b^k}\langle \gamma^k - \alpha^k, \gamma^k - \eta^k\rangle \leq g(\alpha^k, \eta^k) - g(\alpha^k, \gamma^k). \tag{3}$$

By the definition of η^k, we get

$$\frac{1}{b^k}\langle \alpha^k - \eta^k, \gamma - \eta^k\rangle \leq g(\gamma^k, \gamma) - g(\gamma^k, \eta^k), \ \forall \gamma \in \Gamma^k. \tag{4}$$

(3) and (4) imply that

$$\begin{aligned} &\frac{2}{b^k}\langle \alpha^k - \eta^k, \gamma - \eta^k\rangle + \frac{2}{b^k}\langle \gamma^k - \alpha^k, \gamma^k - \eta^k\rangle \\ &\leq 2g(\gamma^k, \gamma) + 2(g(\alpha^k, \eta^k) - g(\alpha^k, \gamma^k) - g(\gamma^k, \eta^k)). \end{aligned} \tag{5}$$

If $g(\alpha^k, \eta^k) - g(\alpha^k, \gamma^k) - g(\gamma^k, \eta^k) > 0$, then

$$g(\alpha^k, \eta^k) - g(\alpha^k, \gamma^k) - g(\gamma^k, \eta^k) \leq c^1\|\alpha^k - \gamma^k\|^2 + c^2\|\eta^k - \gamma^k\|^2. \tag{6}$$

If $g(\alpha^k, \eta^k) - g(\alpha^k, \gamma^k) - g(\gamma^k, \eta^k) \leq 0$, then (6) is obviously true. It follows from (5) and (6) that

$$\begin{aligned} &\langle \alpha^k - \eta^k, \gamma - \eta^k\rangle + \langle \gamma^k - \alpha^k, \gamma^k - \eta^k\rangle \\ &\leq b^k g(\gamma^k, \gamma) + b^k c^1\|\alpha^k - \gamma^k\|^2 + b^k c^2\|\eta^k - \gamma^k\|^2. \end{aligned} \tag{7}$$

Note that

$$\langle \alpha^k - \eta^k, \eta^k - \gamma\rangle = \frac{1}{2}(\|\alpha^k - \gamma\|^2 - \|\alpha^k - \eta^k\|^2 - \|\eta^k - \gamma\|^2) \tag{8}$$

and

$$\langle \alpha^k - \gamma^k, \eta^k - \gamma^k\rangle = \frac{1}{2}(\|\alpha^k - \gamma^k\|^2 + \|\eta^k - \gamma^k\|^2 - \|\alpha^k - \eta^k\|^2). \tag{9}$$

By (7), (8), and (9), we have, $\forall \gamma \in \mathbb{C}$,

$$\begin{aligned} \|\eta^k - \gamma\|^2 \leq{}& 2b^k g(\gamma^k, \gamma) - (1 - 2b^k c^1)\|\alpha^k - \gamma^k\|^2 \\ &- (1 - 2b^k c^2)\|\eta^k - \gamma^k\|^2 + \|\alpha^k - \gamma\|^2. \end{aligned} \tag{10}$$

With $\gamma = \theta^* \in EP(g, \mathbb{C}) \subset \mathbb{C}$, one has $g(\theta^*, \gamma^k) \geq 0, \forall k$. By (A5), we gain $g(\gamma^k, \theta^*) \leq 0, \ \forall k$. Thus, we gain from (10) that

$$\|\eta^k - \theta^*\|^2 \leq \|\alpha^k - \theta^*\|^2 - (1 - 2b^k c^1)\|\alpha^k - \gamma^k\|^2 - (1 - 2b^k c^2)\|\eta^k - \gamma^k\|^2. \quad (11)$$

It follows from $b^k \in (0, \frac{1}{2\max\{c^1, c^2\}})$ and (11) that we have

$$\|\eta^k - \theta^*\| \leq \|\alpha^k - \theta^*\|.$$

On the other hand, we have

$$\|\epsilon^{k+1} - \theta^*\|^2 = (1 - r)\|\alpha^k - \theta^*\|^2 + r\|\eta^k - \theta^*\|^2 - (1 - r)r\|\eta^k - \alpha^k\|^2. \quad (12)$$

By (11) and (12), we have

$$\begin{aligned} \|\epsilon^{k+1} - \theta^*\|^2 \leq{}& \|\alpha^k - \theta^*\|^2 - r\|\alpha^k - \theta^*\|^2 + r\|\alpha^k - \theta^*\|^2 \\ &- r(1 - 2b^k c^1)\|\alpha^k - \gamma^k\|^2 - r(1 - 2b^k c^2)\|\eta^k - \gamma^k\|^2 \\ &- (1 - r)r\|\eta^k - \alpha^k\|^2. \end{aligned} \quad (13)$$

Moreover, we have $\eta^k - \alpha^k = \frac{1}{r}(\epsilon^{k+1} - \alpha^k)$, which, together with (13), gives

$$\begin{aligned} \|\epsilon^{k+1} - \theta^*\|^2 &\leq \|\alpha^k - \theta^*\|^2 - r(1 - 2b^k c^1)\|\alpha^k - \gamma^k\|^2 \\ &\quad - r(1 - 2b^k c^2)\|\eta^k - \gamma^k\|^2 - (1 - r)r\frac{1}{r^2}\|\epsilon^{k+1} - \alpha^k\|^2 \\ &\leq \|\alpha^k - \theta^*\|^2 - \frac{1 - r}{r}\|\epsilon^{k+1} - \alpha^k\|^2 \\ &\leq \|\alpha^k - \theta^*\|^2 - \varsigma\|\epsilon^{k+1} - \alpha^k\|^2, \ \forall k \geq \Lambda, \end{aligned} \quad (14)$$

where $\varsigma = \frac{1-r}{r}$. □

Lemma 3.2. *Assume that Condition 3 and Condition 3 hold. Let $\{\epsilon^k\}$ be a sequence generated by Algorithm 3. Then, for all $\theta^* \in EP(g, \mathbb{C})$,*

$$\begin{aligned} &- 2d^k\langle \epsilon^k - \theta^*, \epsilon^k - G(\epsilon^k)\rangle \\ &\geq \|\epsilon^{k+1} - \theta^*\|^2 - \|\epsilon^k - \theta^*\|^2 + 2n^{k+1}\|\epsilon^{k+1} - \epsilon^k\|^2 - 2n^k\|\epsilon^k - \epsilon^{k-1}\|^2 \\ &\quad + (1 - d^k)n^{k-1}\|\epsilon^{k-1} - \theta^*\|^2 - n^k\|\epsilon^k - \theta^*\|^2 + d^{k+1}\|G(\epsilon^k) - \epsilon^{k+1}\|^2 \\ &\quad - d^k\|\epsilon^k - G(\epsilon^k)\|^2 + (1 - 3n^{k+1} - d^k)\|\epsilon^k - \epsilon^{k+1}\|^2. \end{aligned} \quad (15)$$

Proof. By Lemma 3.1, we get

$$\|\epsilon^{k+1} - \theta^*\|^2 \leq \|\alpha^k - \theta^*\|^2 - \|\epsilon^{k+1} - \alpha^k\|^2. \quad (16)$$

In addition, from the definition of α^k, we gain that

$$\begin{aligned} \|\alpha^k - \theta^*\|^2 &= \|d^k G(\epsilon^k) + (1 - d^k)\beta^k - \theta^*\|^2 \\ &= \|d^k G(\epsilon^k) + (1 - d^k)\epsilon^k + (1 - d^k)n^k(\epsilon^k - \epsilon^{k-1}) - \theta^*\|^2 \\ &= \|\epsilon^k - \theta^*\|^2 + \|(1 - d^k)n^k(\epsilon^k - \epsilon^{k-1}) - d^k(\epsilon^k - G(\epsilon^k))\|^2 \\ &\quad + 2\langle \epsilon^k - \theta^*, (1 - d^k)n^k(\epsilon^k - \epsilon^{k-1}) - d^k(\epsilon^k - G(\epsilon^k))\rangle \\ &= \|\epsilon^k - \theta^*\|^2 + \|(1 - d^k)n^k(\epsilon^k - \epsilon^{k-1}) - d^k(\epsilon^k - G(\epsilon^k))\|^2 \\ &\quad + 2(1 - d^k)n^k\langle \epsilon^k - \theta^*, \epsilon^k - \epsilon^{k-1}\rangle - 2d^k\langle \epsilon^k - \theta^*, \epsilon^k - G(\epsilon^k)\rangle. \end{aligned} \quad (17)$$

Replacing θ^* by ϵ^{k+1} in (17), we have

$$\begin{aligned}\|\alpha^k-\epsilon^{k+1}\|^2 &= \|\epsilon^k-\epsilon^{k+1}\|^2+2(1-d^k)n^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-\epsilon^{k-1}\rangle\\ &\quad-2d^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-G(\epsilon^k)\rangle\\ &\quad+\|(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})-d^k(\epsilon^k-G(\epsilon^k))\|^2.\end{aligned}\tag{18}$$

Substituting (17) and (18) into (16), we have

$$\begin{aligned}&\|\epsilon^{k+1}-\theta^*\|^2\\ &\le \|\epsilon^k-\theta^*\|^2+\|(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})-d^k(\epsilon^k-G(\epsilon^k))\|^2\\ &\quad+2(1-d^k)n^k\langle\epsilon^k-\theta^*,\epsilon^k-\epsilon^{k-1}\rangle-2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle-\|\epsilon^k-\epsilon^{k+1}\|^2\\ &\quad-2(1-d^k)n^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-\epsilon^{k-1}\rangle+2d^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-G(\epsilon^k)\rangle\\ &\quad-\|(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})-d^k(\epsilon^k-G(\epsilon^k))\|^2\\ &= \|\epsilon^k-\theta^*\|^2+2(1-d^k)n^k\langle\epsilon^k-\theta^*,\epsilon^k-\epsilon^{k-1}\rangle-2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle\\ &\quad-\|\epsilon^k-\epsilon^{k+1}\|^2+2d^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-G(\epsilon^k)\rangle+(1-d^k)n^k\|\epsilon^k-\epsilon^{k+1}\|^2\\ &\quad+(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2-(1-d^k)n^k\|\epsilon^k-\epsilon^{k+1}+(\epsilon^k-\epsilon^{k-1})\|^2.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\|\epsilon^{k+1}-\theta^*\|^2-\|\epsilon^k-\theta^*\|^2-(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2+\|\epsilon^k-\epsilon^{k+1}\|^2\\ &-(1-d^k)n^k\|\epsilon^k-\epsilon^{k+1}\|^2\\ &\le 2(1-d^k)n^k\langle\epsilon^k-\theta^*,\epsilon^k-\epsilon^{k-1}\rangle-2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle\\ &\quad+2d^k\langle\epsilon^k-\epsilon^{k+1},\epsilon^k-G(\epsilon^k)\rangle\\ &= -2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle-(1-d^k)n^k\|\epsilon^{k-1}-\theta^*\|^2+(1-d^k)n^k\|\epsilon^k-\theta^*\|^2\\ &\quad+(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2-d^k\|G(\epsilon^k)-\epsilon^{k+1}\|^2+d^k\|\epsilon^k-\epsilon^{k+1}\|^2\\ &\quad+d^k\|\epsilon^k-G(\epsilon^k)\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&-2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle\\ &\ge \|\epsilon^{k+1}-\theta^*\|^2-\|\epsilon^k-\theta^*\|^2-(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2+\|\epsilon^k-\epsilon^{k+1}\|^2\\ &\quad-(1-d^k)n^k\|\epsilon^k-\epsilon^{k+1}\|^2+(1-d^k)n^k\|\epsilon^{k-1}-\theta^*\|^2-(1-d^k)n^k\|\epsilon^k-\theta^*\|^2\\ &\quad-(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2+d^k\|G(\epsilon^k)-\epsilon^{k+1}\|^2\\ &\quad-d^k\|\epsilon^k-\epsilon^{k+1}\|^2-d^k\|\epsilon^k-G(\epsilon^k)\|^2\\ &\ge \|\epsilon^{k+1}-\theta^*\|^2-\|\epsilon^k-\theta^*\|^2+2n^{k+1}\|\epsilon^{k+1}-\epsilon^k\|^2-2(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|^2\\ &\quad+(1-d^k)n^k\|\epsilon^{k-1}-\theta^*\|^2-(1-d^k)n^k\|\epsilon^k-\theta^*\|^2+d^k\|G(\epsilon^k)-\epsilon^{k+1}\|^2\\ &\quad-d^k\|\epsilon^k-G(\epsilon^k)\|^2+(1-(1-d^k)n^k-2n^{k+1}-d^k)\|\epsilon^{k+1}-\epsilon^k\|^2\\ &\ge \|\epsilon^{k+1}-\theta^*\|^2-\|\epsilon^k-\theta^*\|^2+2n^{k+1}\|\epsilon^{k+1}-\epsilon^k\|^2-2n^k\|\epsilon^k-\epsilon^{k-1}\|^2\\ &\quad+(1-d^k)n^k\|\epsilon^{k-1}-\theta^*\|^2-n^k\|\epsilon^k-\theta^*\|^2+d^k\|G(\epsilon^k)-\epsilon^{k+1}\|^2\\ &\quad-d^k\|\epsilon^k-G(\epsilon^k)\|^2+(1-n^k-2n^{k+1}-d^k)\|\epsilon^{k+1}-\epsilon^k\|^2.\end{aligned}$$

As d^k is non-increasing and n^k is non-decreasing, we then gain

$$-2d^k\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle$$

$$\begin{aligned}
&\geq \|\epsilon^{k+1}-\theta^*\|^2-\|\epsilon^k-\theta^*\|^2+2n^{k+1}\|\epsilon^{k+1}-\epsilon^k\|^2-2n^k\|\epsilon^k-\epsilon^{k-1}\|^2\\
&\quad+(1-d^k)n^{k-1}\|\epsilon^{k-1}-\theta^*\|^2-n^k\|\epsilon^k-\theta^*\|^2+d^{k+1}\|G(\epsilon^k)-\epsilon^{k+1}\|^2\\
&\quad-d^k\|\epsilon^k-G(\epsilon^k)\|^2+(1-3n^{k+1}-d^k)\|\epsilon^k-\epsilon^{k+1}\|^2.
\end{aligned}$$

□

Lemma 3.3. *Assume that Condition 3 and Condition 3 hold. Then, $\{\epsilon^k\}$ generated by Algorithm 3 is bounded.*

Proof. By (14) and Condition 3 (i), we can find $\mathtt{B}>0$ such that

$$\begin{aligned}
&\|\epsilon^{k+1}-\theta^*\|\\
&\leq \|\alpha^k-\theta^*\|\\
&= \|d^kG(\epsilon^k)+(1-d^k)\beta^k-\theta^*\|\\
&= \|d^kG(\epsilon^k)+(1-d^k)\epsilon^k+(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})-\theta^*\|\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|+(1-d^k)\|\epsilon^k-\theta^*\|+(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|\\
&= d^k(\|G(\epsilon^k)-G(\theta^*)\|+\|G(\theta^*)-\theta^*\|)+(1-d^k)\|\epsilon^k-\theta^*\|\\
&\quad+(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|\\
&\leq (d^kd+(1-d^k))\|\epsilon^k-\theta^*\|+d^k\|G(\theta^*)-\theta^*\|+d^k\frac{(1-d^k)n^k}{d^k}\|\epsilon^k-\epsilon^{k-1}\|\\
&\leq (1-d^k(1-d))\|\epsilon^k-\theta^*\|+d^k\|G(\theta^*)-\theta^*\|+d^k\mathtt{B}\\
&\leq (1-d^k(1-d))\|\epsilon^k-\theta^*\|+d^k(1-d)(\frac{\|G(\theta^*)-\theta^*\|+\mathtt{B}}{1-d})\\
&\quad\vdots\\
&\leq \max\{\|\epsilon^1-\theta^*\|,\frac{\|G(\theta^*)-\theta^*\|+\mathtt{B}}{1-d}\}.
\end{aligned}$$

This implies that $\{\epsilon^k\}$ is bounded. □

Lemma 3.4. *Suppose that Condition 3 and Condition 3 hold. Let $\{\epsilon^k\}$ be a sequence generated by Algorithm 3. For each $k\geq 1$, define*

$$\chi^k=\|\epsilon^k-\theta^*\|^2-n^{k-1}\|\epsilon^{k-1}-\theta^*\|^2+2n^k\|\epsilon^k-\epsilon^{k-1}\|^2+d^k\|\epsilon^k-G(\epsilon^k)\|^2.$$

Then $\chi^k\geq 0$.

Proof. By the non-decreasingness of $\{n^k\}$ with $0\leq n^k<\frac{1}{3}$, and $2\langle s,v\rangle=\|s\|^2+\|v\|^2-\|s-v\|^2$ for all $s,v\in\mathbb{H}$, we obtain

$$\begin{aligned}
\chi^k &= \|\epsilon^k-\theta^*\|^2-n^{k-1}[\|\epsilon^{k-1}-\epsilon^k\|^2+\|\epsilon^k-\theta^*\|^2+2\langle\epsilon^{k-1}-\epsilon^k,\epsilon^k-\theta^*\rangle]\\
&\quad+2n^k\|\epsilon^k-\epsilon^{k-1}\|^2+d^k\|\epsilon^k-G(\epsilon^k)\|^2\\
&= \|\epsilon^k-\theta^*\|^2-n^{k-1}[2\|\epsilon^{k-1}-\epsilon^k\|^2+2\|\epsilon^k-\theta^*\|^2-\|\epsilon^{k-1}-2\epsilon^k+\theta^*\|^2]\\
&\quad+2n^k\|\epsilon^k-\epsilon^{k-1}\|^2+d^k\|\epsilon^k-G(\epsilon^k)\|^2\\
&\geq \|\epsilon^k-\theta^*\|^2-2n^k\|\epsilon^{k-1}-\epsilon^k\|^2-\frac{2}{3}\|\epsilon^k-\theta^*\|^2+n^{k-1}\|\epsilon^{k-1}-2\epsilon^k+\theta^*\|^2\\
&\quad+2n^k\|\epsilon^k-\epsilon^{k-1}\|^2+d^k\|\epsilon^k-G(\epsilon^k)\|^2\\
&\geq \frac{1}{3}\|\epsilon^k-\theta^*\|^2+d^k\|\epsilon^k-G(\epsilon^k)\|^2
\end{aligned}$$

≥ 0.

□

Lemma 3.5. *Suppose that Condition 3 and Condition 3 hold. Let $\{\epsilon^k\}$ be a sequence generated by Algorithm 3. Suppose*

$$\lim_{k\to\infty} \|\epsilon^{k+1} - \epsilon^k\| = 0,$$

and

$$\lim_{k\to\infty} (\|\epsilon^{k+1} - \theta^*\|^2 - n^k\|\epsilon^k - \theta^*\|^2) = 0.$$

Then $\{\epsilon^k\}$ converges strongly to $\theta^* \in EP(g, \mathbb{C})$.

Proof. By the assumptions, we obtain

$$\lim_{k\to\infty} [(\|\epsilon^{k+1} - \theta^*\| - \sqrt{n^k}\|\epsilon^k - \theta^*\|)(\|\epsilon^{k+1} - \theta^*\| + \sqrt{n^k}\|\epsilon^k - \theta^*\|)] = 0. \tag{19}$$

In the case

$$\lim_{k\to\infty} (\|\epsilon^{k+1} - \theta^*\| + \sqrt{n^k}\|\epsilon^k - \theta^*\|) = 0,$$

we obtain that $\{\epsilon^k\}$ converges strongly to θ^*. Suppose that $\lim_{k\to\infty}(\|\epsilon^{k+1} - \theta^*\| + \sqrt{n^k}\|\epsilon^k - \theta^*\|) \neq 0$. Then there are a set of natural numbers $\mathbb{N}^*$ and $\Phi > 0$ such that

$$\|\epsilon^{k+1} - \theta^*\| + \sqrt{n^k}\|\epsilon^k - \theta^*\| \geq \Phi, \ \forall k \in \mathbb{N}^*. \tag{20}$$

By (19) and the assumption $n^k \leq n < 1$, for $k \in \mathbb{N}^*$ we have

$$\begin{aligned}
0 &= \lim_{k\to\infty} (\|\epsilon^{k+1} - \theta^*\| - \sqrt{n^k}\|\epsilon^k - \theta^*\|) \\
&\geq \limsup_{k\to\infty} (\|\epsilon^k - \theta^*\| - \|\epsilon^{k+1} - \epsilon^k\| - \sqrt{n^k}\|\epsilon^k - \theta^*\|) \\
&\geq \limsup_{k\to\infty} ((1 - \sqrt{n})\|\epsilon^k - \theta^*\| - \|\epsilon^{k+1} - \epsilon^k\|) \\
&= (1 - \sqrt{n}) \limsup_{k\to\infty} \|\epsilon^k - \theta^*\| - \lim_{k\to\infty} \|\epsilon^{k+1} - \epsilon^k\| \\
&= (1 - \sqrt{n}) \limsup_{k\to\infty} \|\epsilon^k - \theta^*\|.
\end{aligned}$$

This implies that $\limsup_{k\to\infty} \|\epsilon^k - \theta^*\| \leq 0$. It is obvious that $\liminf_{k\to\infty} \|\epsilon^k - \theta^*\| \geq 0$, we have $\lim_{k\to\infty} \|\epsilon^k - \theta^*\| = 0$. It follows by (20) that

$$\begin{aligned}
\|\epsilon^{k+1} - \epsilon^k\| &\geq \|\epsilon^{k+1} - \theta^*\| - \|\epsilon^k - \theta^*\| \\
&= \|\epsilon^{k+1} - \theta^*\| + \sqrt{n^k}\|\epsilon^k - \theta^*\| - (1 + \sqrt{n^k})\|\epsilon^k - \theta^*\| \\
&\geq \frac{\Phi}{2},
\end{aligned}$$

for all $k \in \mathbb{N}^*$ sufficiently large, which is a contradiction to $\lim_{k\to\infty} \|\epsilon^{k+1} - \epsilon^k\| = 0$. Thus, the proof is Lemma 3.9 is completed. □

Theorem 3.6. *Suppose that Condition 3 and Condition 3 hold. Then, $\{\epsilon^k\}$ generated by Algorithm 3 strongly converges to the solution $\theta^* = P_{EP(g,\mathbb{C})}G(\theta^*)$.*

Proof. From (15) and Lemma 3.4, we get

$$\begin{aligned}
&\chi^{k+1}-\chi^{k}-n^{k-1}\|\epsilon^{k-1}-\theta^{*}\|^{2}+(1-d^{k})n^{k-1}\|\epsilon^{k-1}-\theta^{*}\|^{2}\\
&-d^{k+1}\|\epsilon^{k+1}-G(\epsilon^{k+1})\|^{2}+d^{k+1}\|G(\epsilon^{k})-\epsilon^{k+1}\|^{2}\\
&+(1-3n^{k+1}-d^{k})\|\epsilon^{k}-\epsilon^{k+1}\|^{2}\\
&\leq -2d^{k}\langle\epsilon^{k}-\theta^{*},\epsilon^{k}-G(\epsilon^{k})\rangle.
\end{aligned}\tag{21}$$

From the Banach fixed point theorem, there exists uniquely $\theta^{*}=P_{EP(g,\mathbb{C})}G(\theta^{*})$. By Lemma 3.1, we have

$$\begin{aligned}
&\|\epsilon^{k+1}-\theta^{*}\|^{2}\\
&\leq\|\alpha^{k}-\theta^{*}\|^{2}\\
&=\|d^{k}(G(\epsilon^{k})-\theta^{*})+(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}\\
&\leq\|(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}\\
&\quad+2\langle d^{k}(G(\epsilon^{k})-\theta^{*}),\alpha^{k}-\theta^{*}\rangle\\
&=\|(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}\\
&\quad+2d^{k}\langle G(\epsilon^{k})-G(\theta^{*}),\alpha^{k}-\theta^{*}\rangle+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\\
&\leq\|(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\\
&\quad+2d^{k}d\|\epsilon^{k}-\theta^{*}\|\|\alpha^{k}-\theta^{*}\|\\
&\leq\|(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\\
&\quad+d^{k}d(\|\epsilon^{k}-\theta^{*}\|^{2}+\|\alpha^{k}-\theta^{*}\|^{2})\\
&\leq\frac{1}{1-d^{k}d}\Big(\|(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\|^{2}+d^{k}d\|\epsilon^{k}-\theta^{*}\|^{2}\\
&\quad+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\Big)\\
&\leq\frac{1}{1-d^{k}d}\Big(\|(1-d^{k})(\epsilon^{k}-\theta^{*})\|^{2}+d^{k}d\|\epsilon^{k}-\theta^{*}\|^{2}\\
&\quad+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\\
&\quad+2\langle(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1}),(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\rangle\Big)\\
&=\frac{(1-d^{k})^{2}+d^{k}d}{1-d^{k}d}\|\epsilon^{k}-\theta^{*}\|^{2}+\frac{1}{1-d^{k}d}\Big(2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\\
&\quad+2\langle(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1}),(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\rangle\Big)\\
&=\left(1-(\frac{2d^{k}(1-d)}{1-d^{k}d}-\frac{(d^{k})^{2}}{1-d^{k}d})\right)\|\epsilon^{k}-\theta^{*}\|^{2}\\
&\quad+\frac{1}{1-d^{k}d}\Big(2\langle(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1}),\\
&\quad(1-d^{k})(\epsilon^{k}-\theta^{*})+(1-d^{k})n^{k}(\epsilon^{k}-\epsilon^{k-1})\rangle+2d^{k}\langle G(\theta^{*})-\theta^{*},\alpha^{k}-\theta^{*}\rangle\Big)
\end{aligned}$$

$$\leq \left(1-\frac{2d^k(1-d)}{1-d^k d}\right)\|\epsilon^k-\theta^*\|^2 + \frac{2d^k(1-d)}{1-d^k d}\left(\frac{d^k}{2(1-d)}\|\epsilon^k-\theta^*\|^2+\frac{1}{d^k(1-d)}\langle(1-d^k)n^k(\epsilon^k-\epsilon^{k-1}), (1-d^k)(\epsilon^k-\theta^*)+(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})\rangle + \frac{1}{1-d}\langle G(\theta^*)-\theta^*, \alpha^k-\theta^*\rangle\right). \tag{22}$$

Next, we shall reconsider into 2 cases. Case 1: If $\chi^{k+1}\leq\chi^k+t^k$ for all $k\geq k^0$ for some $k^0\in\mathbb{N}$, $t^k\geq 0$, and $\sum_{k=1}^{\infty}t^k<+\infty$, by Lemma 2.3 we have that $\lim_{k\to\infty}\chi^k=\lim_{k\to\infty}\chi^{k+1}$ exists. Since $\{\epsilon^k\}$ is bounded by Lemma 3.3, we can find $M_1>0$ such that $2|\langle\epsilon^k-\theta^*,\epsilon^k-G(\epsilon^k)\rangle|+n^{k-1}\|\epsilon^{k-1}-\theta^*\|^2\leq M_1$ and $N_2>0$ such that $\|\epsilon^{k+1}-G(\epsilon^{k+1})\|^2\leq N_2$. Since $0\leq n^k\leq n^{k+1}\leq n<\frac{1}{3}$ and $\lim_{k\to\infty}d^k=0$, there exist $\mathtt{M}\in\mathbb{N}$ and $\varpi^1>0$ such that $1-3n^{k+1}-d^k>\varpi^1$ for all $k\geq\mathtt{M}$. As a result, for $k\geq\mathtt{M}$, we gain from (21) that

$$\varpi^1\|\epsilon^{k+1}-\epsilon^k\|^2\leq d^kM_1+d^{k+1}N_2+\chi^k-\chi^{k+1}\to 0, \tag{23}$$

as $k\to\infty$. Thus, $\lim_{k\to\infty}\|\epsilon^{k+1}-\epsilon^k\|=0$. For $\theta^*\in EP(g,\mathbb{C})$, we have

$$\begin{aligned}
&\|\alpha^k-\theta^*\|^2\\
&=\|d^kg(\epsilon^k)+(1-d^k)\epsilon^k+(1-d^k)n^k(\epsilon^k-\epsilon^{k-1})-\theta^*\|^2\\
&\leq\|d^kG(\epsilon^k)+(1-d^k)\epsilon^k-\theta^*\|^2+2\langle(1-d^k)n^k(\epsilon^k-\epsilon^{k-1}),\alpha^k-\theta^*\rangle\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|^2+(1-d^k)\|\epsilon^k-\theta^*\|^2+2(1-d^k)n^k\|\epsilon^k-\epsilon^{k-1}\|\|\alpha^k-\theta^*\|\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|^2+(1-d^k)\|\epsilon^k-\theta^*\|^2+\frac{2(1-d^k)n^k}{d^k}\|\epsilon^k-\epsilon^{k-1}\|\|\alpha^k-\theta^*\|\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|^2+\|\epsilon^k-\theta^*\|^2+\frac{2(1-d^k)n^k}{d^k}\|\epsilon^k-\epsilon^{k-1}\|\|\alpha^k-\theta^*\|,
\end{aligned}$$

and from (13) we have

$$\begin{aligned}
&\|\epsilon^{k+1}-\theta^*\|^2\\
&=\|\alpha^k-\theta^*\|^2-r(1-2b^kc^1)\|\alpha^k-\gamma^k\|^2-r(1-2b^kc^2)\|\eta^k-\gamma^k\|^2\\
&\quad-(1-r)r\frac{1}{r^2}\|\epsilon^{k+1}-\alpha^k\|^2\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|^2+\|\epsilon^k-\theta^*\|^2+\frac{2(1-d^k)n^k}{d^k}\|\epsilon^k-\epsilon^{k-1}\|\|\alpha^k-\theta^*\|\\
&\quad-r(1-2b^kc^1)\|\alpha^k-\gamma^k\|^2-r(1-2b^kc^2)\|\eta^k-\gamma^k\|^2-\frac{1-r}{r}\|\epsilon^{k+1}-\alpha^k\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
&r(1-2b^kc^1)\|\alpha^k-\gamma^k\|^2+r(1-2b^kc^2)\|\eta^k-\gamma^k\|^2+\frac{1-r}{r}\|\epsilon^{k+1}-\alpha^k\|^2\\
&\leq d^k\|G(\epsilon^k)-\theta^*\|^2+\|\epsilon^k-\theta^*\|^2+\frac{2(1-d^k)n^k}{d^k}\|\epsilon^k-\epsilon^{k-1}\|\|\alpha^k-\theta^*\|\\
&\quad-\|\epsilon^{k+1}-\theta^*\|^2.
\end{aligned}$$

Then, it follows from (23) and our condition that

$$\lim_{k\to\infty}\|\alpha^k-\gamma^k\|=\lim_{k\to\infty}\|\eta^k-\gamma^k\|=\lim_{k\to\infty}\|\epsilon^{k+1}-\alpha^k\|=0. \tag{24}$$

Since $\{\epsilon^k\}$ is bounded, we can find a subsequence $\{\epsilon^{k^i}\}$ of $\{\epsilon^k\}$ such that $\epsilon^{k^i}\rightharpoonup\epsilon^*$ for some $\epsilon^*\in\mathbb{H}$. From (23) and (24), we gain $\alpha^{k^i}\rightharpoonup\epsilon^*$ and $\gamma^{k^i}\rightharpoonup\epsilon^*$ as $i\to\infty$.

By (6) and the definition of η^k, we obtain

$$\begin{aligned}
&b^{k^i}g(\gamma^{k^i},\gamma)\\
&\geq b^{k^i}g(\gamma^{k^i},\eta^{k^i})+\langle\alpha^{k^i}-\eta^{k^i},\gamma-\eta^{k^i}\rangle\\
&\geq b^{k^i}g(\alpha^{k^i},\eta^{k^i})-b^{k^i}g(\alpha^{k^i},\gamma^{k^i})-c^1\|\alpha^{k^i}-\gamma^{k^i}\|^2-c^2\|\eta^{k^i}-\gamma^{k^i}\|^2\\
&\quad+\langle\alpha^{k^i}-\eta^{k^i},\gamma-\eta^{k^i}\rangle\\
&\geq\langle\gamma^{k^i}-\alpha^{k^i},\gamma^{k^i}-\eta^{k^i}\rangle+\langle\alpha^{k^i}-\eta^{k^i},\gamma-\eta^{k^i}\rangle-c^1\|\alpha^{k^i}-\gamma^{k^i}\|^2-c^2\|\eta^{k^i}-\gamma^{k^i}\|^2.
\end{aligned}$$

It follows from this that $\{\eta^{k^i}\}$ is bounded, and from $0<b^{k^i}\leq b<\frac{1}{2\max\{c^1,c^2\}}$ and Condition 3 (A2) that $0\leq\limsup_{i\to\infty}g(\gamma^{k^i},\gamma)\leq g(\epsilon^*,\gamma)\ \forall\gamma\in\mathbb{H}$. These yield that $g(\epsilon^*,\gamma)\geq 0\ \forall\gamma\in\mathbb{C}$, which implies that $\epsilon^*\in EP(g,\mathbb{C})$. Therefore, we gain

$$\begin{aligned}
\limsup_{k\to\infty}\langle G(\theta^*)-\theta^*,\alpha^k-\theta^*\rangle&=\lim_{i\to\infty}\langle G(\theta^*)-\theta^*,\alpha^{k^i}-\theta^*\rangle\\
&=\langle G(\theta^*)-\theta^*,\epsilon^*-\theta^*\rangle\\
&\leq 0,
\end{aligned} \tag{25}$$

by $\theta^*=P_{EP(g,\mathbb{C})}G(\theta^*)$. Using (25) and (22), by Lemma 2.2 we are able to summarize that $\epsilon^k\to\theta^*=P_{EP(g,\mathbb{C})}G(\theta^*)$ as $k\to\infty$.

Case 2: In another case of $\{\chi^k\}$, we give $\Theta:\mathbb{M}\to\mathbb{M}$ as the map defined for all $k\geq k^0$ (for some $k^0\in\mathbb{M}$ large enough) by

$$\Theta(k)=\max\{i\in\mathbb{M}:i\leq k,\chi^i+t^i\leq\chi^{i+1}\}.$$

By the same proof of Case 1 and using (23), we have

$$\varpi^1\|\epsilon^{\Theta(k)+1}-\epsilon^{\Theta(k)}\|^2\leq d^{\Theta(k)}M_1+d^{\Theta(k)+1}N_2+\chi^{\Theta(k)}-\chi^{\Theta(k)+1}\to 0$$

for some constant $M_1,N_2>0$. So,

$$\lim_{k\to\infty}\|\epsilon^{\Theta(k)+1}-\epsilon^{\Theta(k)}\|=0. \tag{26}$$

By the identical proof used in the proof of Case 1, we obtains

$$\lim_{k\to\infty}\|\epsilon^{\Theta(k)+1}-\alpha^{\Theta(k)}\|=\lim_{k\to\infty}\|\alpha^{\Theta(k)}-\epsilon^{\Theta(k)}\|=\lim_{k\to\infty}\|\epsilon^{\Theta(k)}-\eta^{\Theta(k)}\|=0. \tag{27}$$

For $j\geq 0$ by (21), we have $\chi^{j+1}<\chi^j+t^j$ when $\epsilon^j\notin\Omega=\{\epsilon\in\mathbb{H}:\langle\epsilon-G(\epsilon),\epsilon-\theta^*\rangle\leq 0\}$. Hence, $\epsilon^{\Theta(k)}\in\Omega$ for all $k\geq k^0$ since $\chi^{\Theta(k)}+t^{\Theta(k)}\leq\chi^{\Theta(k)+1}$. Since $\{\epsilon^{\Theta(k)}\}$ is bounded, let $\{\epsilon^{\Theta(k)}\}$ be a subsequence of $\{\epsilon^{\Theta(k)}\}$ such that $\epsilon^{\Theta(k)}\rightharpoonup\epsilon^*$, $\epsilon^*\in\mathbb{H}$. Out of Ω is a closed and convex set, so $\epsilon^*\in\Omega$. Using (27), we have $\eta^{\Theta(k)}\rightharpoonup\epsilon^*$. By the same proof in Case 1, we obtain $\epsilon^{\Theta(k)}\rightharpoonup\theta^*=P_{EP(g,\mathbb{C})}G(\theta^*)$. Furthermore,

$$\begin{aligned}
\|\epsilon^{\Theta(k)}-\theta^*\|^2&=\langle\epsilon^{\Theta(k)}-G(\epsilon^k),\epsilon^{\Theta(k)}-\theta^*\rangle-\langle\theta^*-G(\epsilon^k),\epsilon^{\Theta(k)}-\theta^*\rangle\\
&\leq-\langle\theta^*-G(\epsilon^k),\epsilon^{\Theta(k)}-\theta^*\rangle,
\end{aligned}$$

due to $\epsilon^{\Theta(k)} \in \Omega$. This gives

$$\limsup_{k\to\infty} \|\epsilon^{\Theta(k)} - \theta^*\| \leq 0. \tag{28}$$

Accordingly,

$$\lim_{k\to\infty} \|\epsilon^{\Theta(k)} - \theta^*\| = 0.$$

By definition, we have

$$\begin{aligned}\chi^{\Theta(k)+1} &= \|\epsilon^{\Theta(k)+1} - \theta^*\|^2 - n^{\Theta(k)}\|\epsilon^{\Theta(k)} - \theta^*\|^2 + 2n^{\Theta(k)+1}\|\epsilon^{\Theta(k)+1} - \epsilon^{\Theta(k)}\|^2 \\ &\quad + d^{\Theta(k)+1}\|\epsilon^{\Theta(k)+1} - G(\epsilon^{\Theta(k)+1})\|^2 \\ &\leq (\|\epsilon^{\Theta(k)+1} - \epsilon^{\Theta(k)}\| + \|\epsilon^{\Theta(k)} - \theta^*\|)^2 - n^{\Theta(k)}\|\epsilon^{\Theta(k)} - \theta^*\|^2 \\ &\quad + 2n^{\Theta(k)+1}\|\epsilon^{\Theta(k)+1} - \epsilon^{\Theta(k)}\|^2 + d^{\Theta(k)+1}\|\epsilon^{\Theta(k)+1} - G(\epsilon^{\Theta(k)+1})\|^2.\end{aligned}$$

By our Condition 3 (iii), (26), and (28), we gain $\lim_{k\to\infty} \chi^{\Theta(k)+1} = 0$. We shall show that $\lim_{k\to\infty} \chi^k = 0$. For $k \geq k^0$, one has $\chi^k + t^k \leq \chi^{\Theta(k)+1}$ if $k \neq \Theta(k)$, since $\lim_{k\to\infty} t^k = 0$, hence $\limsup_{k\to\infty} \chi^k \leq 0$. By Lemma 3.4 we get $\liminf_{k\to\infty} \chi^k \geq 0$. Hence, we obtain $\lim_{k\to\infty} \chi^k = 0$. It follows from this that $\{\epsilon^k\}$ is bounded, and from $\lim_{k\to\infty} d^k = 0$ and (21) that $\lim_{k\to\infty} \|\epsilon^k - \epsilon^{k+1}\| = 0$. From the definition of χ^k, we have $\lim_{k\to\infty} (\|\epsilon^{k+1} - \theta^*\|^2 - n^k\|\epsilon^k - \theta^*\|^2) = 0$. By Lemma 3.5, $\epsilon^k \to \theta^* = P_{EP(g,\mathbb{C})}G(\theta^*)$. Theorem 3.10 is completed. □

Remark 3.7. (i) The advantage of Algorithm 3 lies in its step size $\{b^k\}$, which can be chosen within the interval $(0, \frac{1}{2\max\{c^1,c^2\}})$. This flexibility makes it particularly suitable for data classification tasks in machine learning, especially when the dataset has a finite number of features.
(ii) The algorithm by Yao et al. [20] features a fixed format for the step size $\{b^k\}$, with a lower bound given by $\min\left\{\frac{\theta\mu}{4\max\{c^1,c^2\}}, b^0\right\}$. This approach is practical without knowing the Lipschitz constants and can be applied to image restoration and signal recovery tasks involving sparse matrices.

4. **Data classification problem.** Cardiovascular diseases or heart diseases are a broader name of conditions that cause abnormalities of the heart and blood vessels in various parts; there are many forms, such as Coronary heart disease, Cerebrovascular disease, Peripheral Arterial Disease, Rheumatic heart disease, Congenital heart disease, Abnormal Heart Rhythms, Deep vein thrombosis, and pulmonary embolism. The leading causes of cardiovascular disease are small lumps of fat gathering together to become plaque, embedded in the blood vessel wall, which blocks transport to various body parts, such as the brain and heart. Cardiovascular diseases are one of the leading causes of death both in Thailand and abroad. In some cases, symptoms may not be present or thought to be a symptom of another disease. This causes the patient not to get the proper treatment and can lead to sudden death. Two factors cause heart disease: those that cannot be controlled (such as genetics, age, and gender) and those that can controlled by lifestyle behaviors (such as smoking, eating high-fat foods, not exercising, and as a result of other diseases for example, diabetes, and high blood pressure). Heart disease patients tend to get

TABLE 1. Data description. There are 3 types of input features:
- Objective: factual information;
- Examination: results of medical examination;
- Subjective: information given by the patient.

Features	Types		
1. Age	Objective Feature	age	int (days)
2. Height	Objective Feature	height	int (cm)
3. Weight	Objective Feature	weight	float (kg)
4. Gender	Objective Feature	gender	categorical code
5. Systolic blood pressure	Examination Feature	ap_hi	int
6. Diastolic blood pressure	Examination Feature	ap_lo	int
7. Cholesterol	Examination Feature	cholesterol	1: normal, 2: above normal, 3: well above normal
8. Glucose	Examination Feature	gluc	1: normal, 2: above normal, 3: well above normal
9. Smoking	Subjective Feature	smoke	binary
10. Alcohol intake	Subjective Feature	alco	binary
11. Physical activity	Subjective Feature	active	binary
12. Presence or absence of cardiovascular disease	Target Variable	cardio	binary

tired quickly when performing various activities, chest pain as if pressed by something, palpitations, and shortness of breath, and swelling of the legs and feet due to heart failure.

Cardiovascular disease is classified as a group of non-communicable diseases (NCDs) that can be prevented by reducing risk factors for disease by changing health behaviors such as blood pressure, fat levels, high sugar levels, BMI, smoking, etc. Cardiovascular diseases (CVDs) are the world's number one killer today. CVDs take the lives of nearly 18 million people every year, or 31% of all global deaths. Triggering these diseases, which manifest primarily as heart attacks and strokes, are tobacco use, unhealthy diet, physical inactivity, and the harmful use of alcohol. These, in turn, show up in people as raised blood pressure, elevated blood glucose, and being overweight or obese, risks detrimental to good heart health. From general information, it is found that most patients are at high risk of developing coronary heart disease because they have aged over 60 years with an average body mass index of $23.9 \pm 4.6\ kg/m^2$, which is a condition of being overweight have atherosclerosis, heart disease with three coronary artery disease is associated with hyperlipidemia and high blood pressure.

In this research, we use the cardiovascular disease dataset which was downloaded from Kaggle (https://www.kaggle.com/datasets/sulianova/cardiovascular-disease-dataset). The dataset is comprised of 70,000 records of patients data, with 1 target and 11 features.

We use the Data Cleaner in MATLAB and use the smooth data technique for height, weight, systolic blood pressure, and diastolic blood pressure. The following are the details of choosing the smoothing factor of parameters.

Table 1, Table 2, and Table 3 show overview of data.

From Table 4, we can see that our Algorithm (MVISE) has higher accuracy compared to the other methods.

TABLE 2. Choices of smoothing method, smoothing parameter, and smoothing factor.

	Smoothing method	Smoothing parameter	Smoothing factor
Height	Moving mean	Smoothing factor	0.25
Weight	Moving mean	Smoothing factor	0.3
Systolic blood pressure (ap_hi)	Moving median	Smoothing factor	0.65
Diastolic blood pressure (ap_lo)	Moving median	Smoothing factor	0.85

TABLE 3. Statistics of all data after preprocessing.

Features	Minimum	Maximum	Mean	Median	Mode	Standard Deviation
Height	105.50	209.00	164.36	164.50	165.00	5.79
Weight	37.00	144.50	74.21	73.00	70.00	10.34
Systolic blood pressure	80.00	210.00	126.13	120.00	120.00	11.95
Diastolic blood pressure	60.00	110.00	81.60	80.00	80.00	4.95

TABLE 4. Accuracy, precision, recall and F-measure of methods in literature and our algorithm.

Methods	Accuracy (%)	Precision (%)	Recall (%)	F-Measure (%)
AdaBoost [2]	73.28	77.10	65.00	70.60
Artificial Neural Network [9]	65.10	63.70	71.80	66.90
Decision Tree [9]	73.10	71.90	76.80	73.90
k-Nearest Neighbors [9]	57.10	57.00	57.80	57.40
LightGBM [2]	74.11	76.30	68.90	72.40
Logistic Regression [9]	72.30	70.30	77.50	73.50
Naive Bayes [9]	59.10	55.50	90.80	68.90
Ontology [9]	75.70	79.30	82.30	80.70
Random Forest [2]	72.08	72.60	69.60	71.10
Support Vector Machine [9]	64.80	63.40	71.20	66.70
XGBoost [2]	74.07	76.30	68.80	72.40
Algorithm of Hieu [18]	74.87	67.99	94.00	78.91
Algorithm of Muangchoo [10]	74.84	67.96	94.03	78.90
Algorithm of Shehu [13]	74.93	68.06	94.00	78.95
Algorithm (VIE) [14]	77.25	71.36	91.06	80.02
Our Algorithm (MVISE)	**77.50**	**71.87**	**90.41**	**80.08**

For our application in machine learning, we use *Extreme Learning Machine* (ELM), which was introduced by Huang et al. [6]. This concept was explained by letting $E := \{(\mathbf{s}_n, \mathbf{r}_n) : \mathbf{s}_n \in \mathbb{R}^n, \mathbf{r}_n \in \mathbb{R}^q, n = 1, 2, 3, ..., \iota\}$ be a training set where $\mathbf{r}_n$ is the target data and $\mathbf{s}_n$ is input data. The goal of the ELM is to find an output function in the form

$$\mathbf{Q}_n = \sum_{\kappa=1}^{M} \Xi_\kappa Z(c_\kappa \mathbf{s}_n + d_\kappa),$$

where M are the hidden nodes, Z is the activation function, c_κ is start random weight, d_κ is the initial random bias, and Ξ_κ is the optimal output weight at the

κ-th node. We can define the hidden layer matrix as follows:

$$L = \begin{bmatrix} Z(c_1\mathbf{s}_1 + d_1) & \dots & Z(c_M\mathbf{s}_1 + d_M) \\ \vdots & \ddots & \vdots \\ Z(c_1\mathbf{s}_\iota + d_1) & \dots & Z(c_M\mathbf{s}_\iota + d_M) \end{bmatrix}.$$

To solve ELM is to find the optimal output weights $\Xi = [\Xi_1^T, ..., \Xi_M^T]^T$ such that $L\Xi = T$, where $T = [\mathbf{r}_1^T, ..., \mathbf{r}_\iota^T]^T$ is the training data. In some cases, one can find $\Xi = L^\dagger T$, where $L^\dagger$ is the *Moore-Penrose generalized inverse* of L. When $L^\dagger$ is difficult to calculate, we can solve $L\Xi = T$ by the least square problems

$$\min_{\Xi \in \mathbb{R}^M} \{\|L\Xi - T\|^2\}, \tag{29}$$

For applying our algorithm to solve (29), we set $f(\Xi, \zeta) = \langle L^T(L\Xi - T), \zeta - \Xi\rangle$.

In this research, we use the binary cross entropy loss function

$$Loss = -\frac{1}{\iota}\sum_{\kappa=1}^{\iota} z_\kappa \log \hat{z}_\kappa + (1 - z_\kappa)\log(1 - \hat{z}_\kappa)$$

where $\hat{z}_\kappa$ is the κ-th scalar value in the model output, ι is the number of samples, and z_κ is the corresponding target value.

Precision defines the potential of positive prediction, recall defines the model's ability to detect pertinent data, and accuracy denotes the rate of correct predictions by the model. The formula of three measures [16] are defined as follows:

$$\begin{aligned} \text{Precision(Pre)} &= \frac{\mathcal{TP}}{\mathcal{TP} + \mathcal{FP}} \times 100\%, \\ \text{Recall(Rec)} &= \frac{\mathcal{TP}}{\mathcal{TP} + \mathcal{FN}} \times 100\%, \\ \text{Accuracy(Acc)} &= \frac{\mathcal{TP} + \mathcal{TN}}{\mathcal{TP} + \mathcal{FP} + \mathcal{TN} + \mathcal{FN}} \times 100\%, \end{aligned}$$

where $\mathcal{TP}$ are true positives, $\mathcal{TN}$ are true negatives, $\mathcal{FP}$ are false positives, and $\mathcal{FN}$ are false negatives. Similarly, $\mathcal{P}$ and $\mathcal{N}$ are the positive and negative populations of malignant and benign cases, respectively. And, we define the F-measure as the harmonic mean value between recall and precision as follows:

$$\text{F-Measure} = 2\left(\frac{\text{Precision} \times \text{Recall}}{\text{Precision} + \text{Recall}}\right) \times 100\%.$$

In our calculation we start at $V(x) = \frac{1}{1+e^{-x}}, M = 90, N = 200, r = 0.5$, $d^k = \frac{k}{1.2k}$, $n = 0.33, and b = 1 \times 10^{-5}$ for Algorithm **(MVISE)** and Algorithm **(MHISE)**, and $D = 0.9999$ for Algorithm **(MVISE)**. We gain the outcomes of the different parameters W where $b^k = \frac{W}{max(eigenvalue(A^TA))}$ for Algorithm **(MVISE)** and Algorithm **(MHISE)** as seen in Table 5.

When

$$n^k = \begin{cases} n & \text{otherwise,} \\ \frac{n}{k^2\|x^k - x^{k-1}\|} & \text{if } k > \mathtt{N} \text{ and } x^k \neq x^{k-1}, \end{cases}$$

we require to stop with N, when N is an iteration number.

We can see that $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$ greatly improves the performance of Algorithm **(MVISE)** and Algorithm **(MHISE)**.

We next select $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $d^k = \frac{k}{1.2k}$, and $r = 0.5$ for Algorithm **(MHISE)**, and $D = 0.9999$ for Algorithm **(MVISE)** with $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$,

TABLE 5. Training time and numerical results of both of training and validation loss when b^k is different.

			Loss	
	W	Training Time	Training	Validation
Algorithm **(MVISE)**	0.2	1.9910	0.2893	0.2895
	0.4	1.9914	0.2802	0.2807
	0.6	1.9977	0.2731	0.2737
	0.9	1.9954	0.2652	0.2660
	0.999	1.9997	0.2632	0.2640
Algorithm **(MHISE)**	0.2	1.9982	0.2943	0.2945
	0.4	1.9985	0.2885	0.2888
	0.6	1.9961	0.2835	0.2839
	0.9	1.9953	0.2771	0.2776
	0.999	1.9933	0.2752	0.2758

TABLE 6. Training time and numerical results of both of training and validation loss when n is different.

			Loss	
	n	Training Time	Training	Validation
Algorithm **(MVISE)**	0.1	1.9950	0.2639	0.2647
	0.15	2.0127	0.2638	0.2646
	0.2	1.9961	0.2636	0.2644
	0.3	1.9957	0.2633	0.2641
	0.33	1.9915	0.2632	0.2640
Algorithm **(MHISE)**	0.1	2.0010	0.2756	0.2762
	0.15	2.0045	0.2755	0.2761
	0.2	1.9990	0.2754	0.2760
	0.3	1.9945	0.2753	0.2758
	0.33	1.9962	0.2752	0.2758

$d^k = \frac{k}{1.2k}$, and $r = 0.5$. With $N = 200$, we investigate the initialization parameter n as seen in Table 6.

We can see that $n = 0.33$ greatly improves the performance of Algorithm **(MVISE)** and Algorithm **(MHISE)**.

We next set $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $n^k = 0.33$, and $r = 0.5$ for Algorithm **(MHISE)**, and $D = 0.9999$ for Algorithm **(MVISE)** with $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $n^k = 0.33$, and $r = 0.5$. With $N = 200$, we investigate the parameter d^k. The numeric outcomes are displayed in Table 7.

We can see that $d^k = \frac{k}{1.2k}$ greatly improves the efficiency of Algorithm **(MVISE)**, and $d^k = \frac{k}{1.2k}$ greatly improves the efficiency of Algorithm **(MHISE)**. As a result, we select it as the default inertial parameter for the next calculation.

TABLE 7. Training time and numerical results of both of training and validation loss when d^k is different

	d^k	Training Time	Loss	
			Training	Validation
Algorithm **(MVISE)**)	$\frac{k}{k+1}$	1.9948	0.2642	0.2649
	$\frac{k}{k+2}$	1.9891	0.2640	0.2648
	$\frac{k}{k+5}$	1.9950	0.2637	0.2645
	$\frac{k}{1.1k}$	1.9910	0.2637	0.2645
	$\frac{k}{1.2k}$	1.9969	0.2632	0.2640
Algorithm **(MHISE)**	$\frac{k}{k+1}$	1.9994	0.2770	0.2775
	$\frac{k}{k+2}$	2.0049	0.2768	0.2773
	$\frac{k}{k+5}$	1.9956	0.2761	0.2766
	$\frac{k}{1.1k}$	1.9922	0.2762	0.2767
	$\frac{k}{1.2k}$	1.9961	0.2752	0.2758

TABLE 8. Training time and numerical results of both of training and validation loss when r is different.

	r	Training Time	Loss	
			Training	Validation
Algorithm **(MVISE)**	0.1	1.9950	0.2894	0.2896
	0.2	1.9946	0.2803	0.2808
	0.3	1.9921	0.2732	0.2738
	0.4	1.9934	0.2676	0.2683
	0.5	1.9986	0.2632	0.2640
Algorithm **(MHISE)**	0.1	1.9967	0.2944	0.2945
	0.2	2.0020	0.2886	0.2889
	0.3	2.0053	0.2835	0.2839
	0.4	1.9927	0.2791	0.2796
	0.5	1.9995	0.2752	0.2758

Next, we compute the numerical outcomes by setting $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $n^k = 0.33$, $d^k = \frac{k}{1.2k}$, and $D = 0.9999$ for Algorithm **(MVISE)** and $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $n^k = 0.33$, and $d^k = \frac{k}{1.2k}$ for Algorithm **(MHISE)**. With $N = 200$, we investigate the different parameter r. The numeric outcomes are dispalyed in Table 8.

We can see that $r = 0.5$ greatly improves the performance of Algorithm **(MVISE)** and Algorithm **(MHISE)**. As a result, we select it as the default parameter for the next calculation.

We calculated the numerical outcomes by setting $b^k = \frac{0.999}{max(eigenvalue(A^TA))}$, $n^k = 0.33$, $r = 0.5$, and $d^k = \frac{k}{1.2k}$ for Algorithm **(MVISE)** and Algorithm **(MHISE)**.

TABLE 9. Training time and numerical results of both of training and validation loss when D is different.

			Loss	
	D	Training Time	Training	Validation
Algorithm **(MVISE)**	0.3	2.009	0.3005	0.3005
	0.5	1.9867	0.3003	0.3003
	0.7	1.9919	0.2998	0.2998
	0.9	1.9978	0.2975	0.2975
	0.9999	1.9980	0.2632	0.2640

TABLE 10. Selected parameters of each algorithm.

	e	b^1	b^k	n^k	d^k	r	D	δ	μ	θ
Algorithm of Hieu [18]	-	-	$\frac{0.999}{2(max(eig(A^TA)))}$	-	$\frac{k}{1.2k}$	-	-	-	-	-
Algorithm of Muangchoo [10]	$\frac{0.999}{max(eig(A^TA))}$	$\frac{0.999}{2(max(eig(A^TA)))}$	-	-	$\frac{k}{1.2k}$	-	-	-	-	-
Algorithm of Shehu [13]	0.9	$\frac{0.999}{2(max(eig(A^TA)))}$	-	0.33	$\frac{k}{1.2k}$	0.5	-	-	-	-
Algorithm (VIE) [14]	-	-	$\frac{0.999}{2(max(eig(A^TA)))}$	0.33	$\frac{k}{1.2k}$	0.5	0.999	-	-	-
Algorithm (MVISE)	-	-	$\frac{0.999}{max(eig(A^TA))}$	0.33	$\frac{k}{1.2k}$	0.5	0.999	-	-	-
Algorithm (MHISE)	-	-	$\frac{0.999}{max(eig(A^TA))}$	0.33	$\frac{k}{1.2k}$	0.5	-	-	-	-
Algorithm of Yao et al. [20]	-	$\frac{0.999}{2(max(eig(A^TA)))}$	-	-0.1	-	-	-	$\frac{\sqrt{5}-1}{2}$	0.1	0.1

TABLE 11. Efficiency comparative of existing algorithms and our algorithms.

Algorithms	**Iteration count**	**Training Time**	**Pre**	**Rec**	**Acc**	**F-measure**
Algorithm of Hieu [18]	52	0.5316	67.99	94.00	74.87	78.91
Algorithm of Muangchoo [10]	52	3.1999	67.96	94.03	74.84	78.90
Algorithm of Shehu [13]	52	1.6957	68.06	94.00	74.93	78.95
Algorithm (VIE) [14]	52	0.5312	71.36	91.06	77.25	80.02
Algorithm (MVISE)	52	0.5341	71.87	90.41	77.50	80.08
Algorithm (MHISE)	52	0.5377	71.86	90.42	77.49	80.08
Algorithm of Yao et al. [20]	52	2.0370	67.53	94.30	74.47	78.70

With $N = 200$, we gain the outcomes of the different parameters D when $G(x) = Dx$ for Algorithm **(MVISE)** as seen in Table 9.

From Tables 5-9, we select the parameters for Algorithm **(MVISE)** to collate the existent algorithms from the introduction.

Table 10 displays the selected essential parameters of each algorithm. For comparison, we set $V(x) = \frac{1}{1+e^{-x}}$, $M = 90$, and $b = 1 \times 10^{-5}$.

Table 11 shows that the Algorithm **(MVISE)** has the highest performance of precision accuracy, and F-measure by the number of iterations is 52. It is the highest possibility of exactly classifying comparisons to algorithms investigations. Both the training-validation loss and accuracy plots show that the model for the cardiovascular disease dataset has a good fitting model when we use the Algorithm **(MVISE)**.

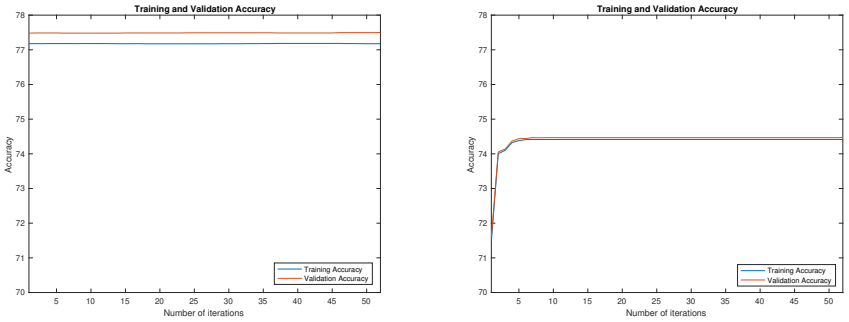

FIGURE 1. On the left are the accuracy plots of training and validation of Algorithm **(MVISE)**, and on the right are the accuracy plots of training and validation of the algorithm of Yao et al. [20].

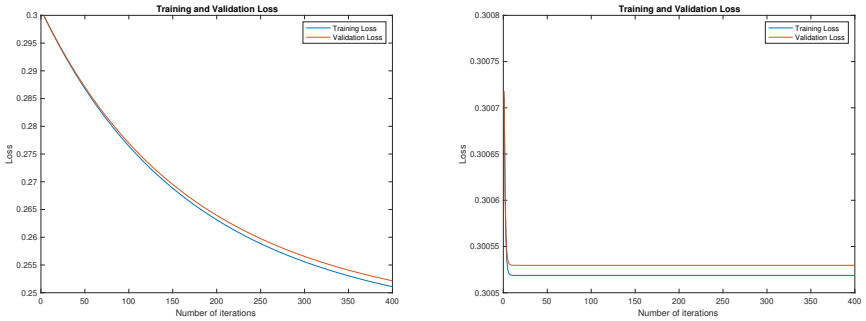

FIGURE 2. On the left are the loss plots of training and validation of algorithm **(MVISE)** and on the right are the loss plots of training and validation of the algorithm of Yao et al. [20].

From Figures 1-2, we see that our Algorithm **(MVISE)**, has a good fit model. This means that our Algorithm **(MVISE)** appropriately learns the training dataset and generalizes well to classify the cardiovascular disease dataset.

5. **Conclusion.** This work presents a viscosity type inertial subgradient extragradient algorithm for nonmonotone equilibrium problems in real Hilbert spaces. The algorithm can be reduced to the Halpern inertial subgradient extragradient algorithm by setting the contraction mapping. The strong convergence theorem is proved under some suitable conditions. In application, selecting the best parameters and showing for our algorithms to get high accuracy to predict cardiovascular

disease classification in machine learning. In addition, we compare our algorithm and the existing works in the literature to show the algorithm's efficiency. The accuracy of our work is 77.50%, which can be used for exactly predicting cardiovascular disease in the future.

Data availability. The data are available in Kaggle (https://www.kaggle.com/datasets/sulianova/cardiovascular-disease-dataset).

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NEW ASYMPTOTICALLY ISOMETRIC PROPERTIES THAT IMPLY THE FAILURE OF THE FIXED POINT PROPERTY IN COPIES OF ℓ^1

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Abstract. In this study, we introduce three new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for "alternative asymptotically isometric". We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an asymptotically isometric copy of ℓ^1. That is, a Banach space contains an asymptotically isometric copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. We also support our ideas with some examples and remarks for our equivalent properties to contain asymptotically isometric copy of ℓ^1.

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1. Preliminaries and Introduction

A Banach space is said to have the fixed point property for nonexpansive mappings [fpp(ne)] if every self-map of any closed, bounded and convex domain in that space satisfying the condition of nonexpansiveness has a fixed point (see for example page 37 in [8]). Here, by nonexpansive map we mean a map $T : C \to C$ such that $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$.

It is a fact that either c_0 or ℓ^1 is almost isometrically embedded in any nonreflexive Banach spaces with an unconditional basis (see e.g. [9]). Thus, all of the classical nonreflexive spaces fail fpp(ne); that is, there exists a closed, bounded and convex subset and a nonexpansive self-map T defined on that set such that T is fixed point free. This result depends on well-known facts (Theorems 1.c.12 in [12] and 1.c.5 in

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[13]) stated by the following: a Banach lattice or a Banach space with an unconditional basis is reflexive if and only if it contains no isomorphic copies of c_0 or ℓ^1. Hence, if it can be shown that neither c_0 nor ℓ^1 can be renormed to have fpp(ne), it would follow that fpp(ne) in either a Banach lattice or in a Banach space with an unconditional basis would imply reflexivity.

In this matter, for many years, researchers have asked the question whether or not either ℓ^1 or c_0 can be equivalently renormed so that the renormed space has fpp(ne). For ℓ^1, there is a fact suggesting the contrary by Lin [11].

In [9], James showed that if a Banach space contains an isomorphic copy of ℓ^1 (respectively, c_0), then it contains almost isometric copies of ℓ^1 (respectively, c_0) and then he provided a tool that helped researches investigate the question of whether ℓ^1 or c_0 can be renormed to have fpp(ne). Using this tool and strengthening that, Dowling, Lennard and Turett, in several articles, have inquired relations between spaces containing nice copies of c_0 or ℓ^1 and the failure of fpp(ne). After strengthening James Distortion Theorems, they investigated these structures and then worked on the notion of a Banach space containing an asymptotically isometric (ai) copy of ℓ^1 or c_0 which are used as important tools in identifying failure of fpp(ne) for Banach spaces. It can be said that at first, their aim of introducing the notion of Banach spaces containing ai copies of ℓ^1 was to prove the fact that nonreflexive subspaces of $L^1[0,1]$ fail fpp(ne). This result can be seen in [3] by Dowling and Lennard.

Later, Dowling, Lennard and Turett [6] showed that if a Banach space contains an ai c_0 copy, then it fails fpp(ne). They also showed in [5] that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1. Furthermore, in [6], they prove that there exists a Banach space failing fpp(ne) whereas it does not contain any ai copy of c_0. In fact, in 2010 [7], they show that any infinite dimensional subspace of their example of asymptotically isometric c_0 free space fails fpp(ne). Hence, one can inquire of existence of a property which may strengthen the notion of ai copy of c_0 so that it can help see failure of fpp(ne) in the case of lack of ai c_0 copy.

For example, in order to obtain alternative methods for detecting failure of fpp(ne), in 2011, Lennard and Nezir [10] proved that if a Banach space contains an ai c_0-summing basic sequence, it fails fpp(ne). In fact, they showed that closed convex hull of any ai c_0-summing basic sequence fails fpp(ne). So there was another tool of detecting spaces failing fpp(ne). Moreover, recently, a nice property was developed by Álvaro, Cembranos and Mendoza [1] to detect failure of fpp(ne). They introduced a new property, N1, that implies an isomorphic copy of c_0 fails fpp(ne) such that their property is more general than the property of Banach spaces to contain ai copies of c_0.

Similarly to their works for Banach spaces containing ai copies of c_0, starting in 1996, Dowling, Lennard and Turett explored Banach spaces containing ai copies of ℓ^1 and they obtained important results leading researchers to test failure of fpp(ne) in

Banach spaces they work on. For example, it can be said that Lin was inspired by the norm developed by Dowling, Johnson, Lennard and Turett [2] when they showed ℓ^1 with that norm does not contain any ai copy of ℓ^1 whereas Lin later proved in [11] that ℓ^1 considered with a special case of the norm has fpp(ne). As there are interests to search alternative properties for detecting asymptotically isometric copy of c_0, it must be significant to find better or even equivalent property for a Banach space to contain an ai copy of ℓ^1.

Hence, aiming to find different tools to detect failure of fpp(ne) in Banach spaces, in this study, we introduce new notions which may occur for some Banach spaces. We call these new properties AAI1, AAI2 and AAI3 where AAI stands for "alternative asymptotically isometric". We prove that if a Banach space has any of them, then it fails to have the fixed point property for nonexpansive mappings [fpp(ne)]. We provide alternative ways of detecting if a Banach space has any of these properties. We show that AAI1 is an equivalent property for a Banach space to have an ai copy of ℓ^1. That is, a Banach space contains an ai copy of ℓ^1 if and only if it has the property AAI1. In fact, we obtain generalized version of property AAI1, which we call property AAI3, obtained as a conclusion of property AAI2. We prove that all properties AAI1, AAI2 and AAI3 are equivalent. Moreover, we support our ideas with some examples and remarks for our equivalent properties to contain ai copy of ℓ^1. For example, we demonstrate an example of a Banach space satisfying property AAI1 where we show that the sequence allowing us to have the property is also an ai ℓ^1 sequence. Furthermore, as our final example, which is an obvious remark due to our result, we give an example of a Banach space failing AAI1 property.

We need to note that in [14], Nezir introduced another alternative asymptotically isometric property inside copies of ℓ^1 but our properties AAI1, AAI2 and AAI3 are still more general than that.

Now, we give some preliminaries required for our work.

Throughout the paper our scalar field is $\mathbb{R}$, c_0 represents the Banach space of scalar sequences converging to 0, ℓ^1 is for the Banach space of absolutely summable sequences, and c_{00} is for the space of sequences of finitely nonzero terms. Furthermore, in the paper, we will be using the unconditional basis $(e_n)_{n\in\mathbb{N}}$ for both $(c_0, \|\cdot\|_\infty)$ and $(\ell^1, \|\cdot\|_1)$ such that it has 1 in its nth coordinate, and 0 in its other coordinates for each $n \in \mathbb{N}$.

Firstly, we will remind the definition of an ai ℓ^1 sequence and we will give the theorem providing alternative way of detecting if there is any as in [4] by Dowling, Lennard and Turett.

Definition 1. [4] We call a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \leq \left\|\sum_{n=1}^{\infty} a_n x_n\right\| \leq \sum_{n=1}^{\infty}|a_n|\ , \text{ for all } (a_n)_n \in \ell^1.$$

Theorem 1. *[4] A Banach space $(X,\|\cdot\|)$ contains an ai copy of ℓ^1 if and only if there is a sequence $(x_n)_n$ in X and there are constants $0<m<M<\infty$ such that for all $(t_n)_n\in\ell^1$,*

$$m\sum_{n=1}^{\infty}|t_n|\leq\left\|\sum_{n=1}^{\infty}t_nx_n\right\|\leq M\sum_{n=1}^{\infty}|t_n|$$

and $\lim_{n\to\infty}\|x_n\|=m$.

Then, the following theorem is given as a result in [3].

Theorem 2. *If a Banach space $(X,\|\cdot\|)$ contains an ai copy of ℓ^1, then X fails fpp(ne).*

2. Alternative Asymptotically Isometric Properties

In this section, we will introduce our AAI properties that will provide alternatives for well-known notions of ai copy of ℓ^1.

Definition 2. We will say that a Banach space $(X,\|\cdot\|)$ has property AAI1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2+\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2}\leq\left\|\sum_{n=1}^{\infty}t_nx_n\right\|\leq\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2},$$

for all $(t_n)_n\in\ell^1$.

Definition 3. We will say that a Banach space $(X,\|\cdot\|)$ has property AAI2 if there exist a sequence $(x_n)_n$ in X, a null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n\in\ell^1\setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x|||=\sum_{k=1}^{\infty}|\rho_k\xi_k|$ for any $x=(\xi_k)_k\in c_0$ so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|+\rho\left(((1-\varepsilon_n)t_n)_n\right)\leq\left\|\sum_{n=1}^{\infty}t_nx_n\right\|\leq\sum_{n=1}^{\infty}|t_n|+\rho\left((t_n)_n\right),$$

for all $(t_n)_n\in\ell^1$.

Definition 4. We will say that a Banach space $(X,\|\cdot\|)$ has property AAI3 if there exist a sequence $(x_n)_n$ in X, a null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n\in\ell^1\setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x|||=\sum_{k=1}^{\infty}|\rho_k\xi_k|$ for any $x=(\xi_k)_k\in$

c_0 so that

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2+[\rho(((1-\varepsilon_n)t_n)_n)]^2}\le\left\|\sum_{n=1}^{\infty}t_nx_n\right\|\le\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+[\rho((t_n)_n)]^2},$$

for all $(t_n)_n\in\ell^1$.

In this section, we explore alternative properties to the notion of ai ℓ^1 copy. We will see that our properties will imply the failure of fpp(ne) in copies of ℓ^1. We prove that each of AAI1, AAI2 and AAI3 is equivalent property to have ai copy of ℓ^1. Then, in the final subsection, we give an example of a Banach space containing an ai ℓ^1 sequence such that the same sequence satisfies the condition of property AAI1. Furthermore, we show an example of a Banach space without AAI1 property or any ai ℓ^1 sequence by a quick remark since we know that a Banach space's failure of having property AAI1 is equivalent to failure of containing an ai copy of ℓ^1.

2.1. *Property AAI1*

In this section, we explore property AAI1 and show that property AAI1 implies having ai copy of ℓ^1 and the converse is also true: that is, both properties imply each other.

First, we give an alternative way of detecting our property AAI1 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI1.

Theorem 3. *A Banach space $(X,\|\cdot\|)$ has property AAI1 if and only if there is a sequence $(x_n)_n$ in X such that there exists $R\in[1,\infty)$ so that for any $(t_n)_n\in\ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}\le\left\|\sum_{n=1}^{\infty}t_nx_n\right\|\le R\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}\tag{2.1}$$

and

$$\lim_{n\to\infty}\|x_n\|=1.\tag{2.2}$$

Proof. Suppose that $(X, \|\cdot\|)$ has property AAI1. Then, there exist a null sequence $(\varepsilon_n)_n$ in $(0,1)$ and a sequence $(x_n)_n$ in X so that

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2+\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty}t_nx_n\right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}, \tag{2.3}$$

for all $(t_n)_n \in \ell^1$.

We may assume $(\varepsilon_n)_{n\in\mathbb{N}}$ to be a decreasing sequence since we may replace that with $\zeta_n := \max_{j\geq n}\varepsilon_j$, for all $n\in\mathbb{N}$. Let $y_n = (1-\varepsilon_n)^{-1}x_n$ for each $n\in\mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty}t_ny_n\right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}\frac{|t_n|}{1-\varepsilon_n}\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{(1-\varepsilon_n)2^n}\right)^2}.$$

Let $R=\frac{1}{1-\varepsilon_1}$. Then, condition (2.1) is satisfied for the sequence $(y_n)_n$ in X. Also, it is easy to check (2.2) for the sequence $(y_n)_n$ too since in inequality (2.3), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 we obtain that $\lim_{n\to\infty}\|x_n\|=1$ and so $\lim_{n\to\infty}\|y_n\|=1$.

Conversely, assume that there exist a sequence $(x_n)_n$ in X and $R\in[1,\infty)$ so that for all $(t_n)_n\in\ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty}t_nx_n\right\| \leq R\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \tag{2.4}$$

and $\lim_{n\to\infty}\|x_n\|=1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim_{n\to\infty}\|x_n\|=1$, and $\|x_n\|\geq 1$ for all $n\in\mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1\leq\|x_n\|\leq 1+\varepsilon_n$ for all $n\in\mathbb{N}$. Define $y_n=\frac{x_n}{1+\varepsilon_n}$ for all $n\in\mathbb{N}$. Then, since $\|y_n\|\leq 1$, we have

$$\left\|\sum_{n=1}^{\infty}t_ny_n\right\| \leq \sum_{n=1}^{\infty}|t_n| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}, \text{ for all } (t_n)_n\in\ell^1.$$

Also, from the left hand side inequality of (2.4), we have

$$\left\|\sum_{n=1}^{\infty} t_n y_n\right\| = \left\|\sum_{n=1}^{\infty} t_n \frac{x_n}{(1+\varepsilon_n)}\right\| \geq \sqrt{\left(\sum_{n=1}^{\infty} \frac{|t_n|}{1+\varepsilon_n}\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{(1+\varepsilon_n)2^n}\right)^2}$$
$$\geq \sqrt{\left(\sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n|\right)^2 + \left(\sum_{n=1}^{\infty} (1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2}.$$

□

Now, we show property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1.

Theorem 4. *Let $(X, \|.\|)$ be a Banach space. Then, X contains an ai copy of ℓ^1 if and only if X has property AAI1.*

Proof. Assume that a Banach space X contains an ai copy of ℓ^1. Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim_{n\to\infty} \|x_n\| = 1$ and $M \in [1,\infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\|\sum_{n=1}^{\infty} t_n x_n\right\| \leq M \sum_{n=1}^{\infty} |t_n|.$$

Then, letting $y_n = \left(1+\frac{1}{2^n}\right) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{3M}{2}$, we have

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \leq \sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n}$$
$$\leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq M \left(\sum_{n=1}^{\infty} |t_n| + \sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right).$$

Hence,

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty} t_n y_n\right\| \leq R \sum_{n=1}^{\infty} |t_n|$$
$$\leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left(\sum_{n=1}^{\infty} \frac{|t_n|}{2^n}\right)^2}$$

and $\lim_n \|y_n\| = 1$.

Thus, by the previous theorem, X has property AAI1.

Conversely assume that X has property AAI1. Then, there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty}\|x_n\| = 1$ and a constant $R \in [1,\infty)$ so that for any $(t_n)_n \in \ell^1$,

$$\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2} \leq \left\|\sum_{n=1}^{\infty}t_n x_n\right\| \leq R\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}$$

and so

$$\sum_{n=1}^{\infty}|t_n| \leq \left\|\sum_{n=1}^{\infty}t_n x_n\right\| \leq \frac{3R}{2}\sum_{n=1}^{\infty}|t_n|\,.$$

Hence, we are done by Theorem 1. □

2.2. *A more generalized property, property AAI2*

Firstly, we give an alternative way of detecting our property AAI2 which will help us prove that a Banach space contains an ai copy of ℓ^1 if and only if it has property AAI2.

Theorem 5. *A Banach space $(X,\|\cdot\|)$ has property AAI2 if and only if there exist a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty}|\rho_k\xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1,\infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sum_{n=1}^{\infty}|t_n| + \rho\left((t_n)_n\right) \leq \left\|\sum_{n=1}^{\infty}t_n x_n\right\| \leq R\left[\sum_{n=1}^{\infty}|t_n| + \rho\left((t_n)_n\right)\right] \tag{2.5}$$

and

$$\lim_{n\to\infty}\|x_n\| = 1. \tag{2.6}$$

Proof. Suppose that $(X,\|\cdot\|)$ has property AAI2. Then, there exist a sequence $(x_n)_n$ in X, a decreasing null sequence $(\varepsilon_n)_n$ in $(0,1)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty}|\rho_k\xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n| + \rho\left(((1-\varepsilon_n)t_n)_n\right) \leq \left\|\sum_{n=1}^{\infty}t_n x_n\right\| \leq \sum_{n=1}^{\infty}|t_n| + \rho\left((t_n)_n\right), \tag{2.7}$$

for all $(t_n)_n \in \ell^1$.

Let $y_n = (1-\varepsilon_n)^{-1}x_n$ for each $n \in \mathbb{N}$. Then, for all $(t_n)_n \in \ell^1$, taking ρ being a lattice norm into consideration,

$$\sum_{n=1}^{\infty}|t_n| + \rho\left((t_n)_n\right) \leq \left\|\sum_{n=1}^{\infty}t_n y_n\right\| \leq \sum_{n=1}^{\infty}\frac{|t_n|}{1-\varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1-\varepsilon_n}\right)_n\right).$$

Let $R = \frac{1}{1-\varepsilon_1}$. Then, condition (2.5) is satisfied for the sequence $(y_n)_n$ in X. Also, it is easy to check (2.6) for the sequence $(y_n)_n$ too since in inequality (2.7), taking $(t_n)_n$ as the unit basis $(e_n)_n$ of c_0 and recalling ρ is equivalent to $|||.|||$, we obtain that $\lim_{n\to\infty} \|x_n\| = 1$ and so $\lim_{n\to\infty} \|y_n\| = 1$.

Conversely, assume that there exist $R \in [1,\infty)$, a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \leq \left|\left|\left| \sum_{n=1}^{\infty} t_n x_n \right|\right|\right| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \right] \tag{2.8}$$

and $\lim_{n\to\infty} \|x_n\| = 1$.

Let $(\varepsilon_n)_n$ be a null sequence in $(0,1)$. Since $\lim_{n\to\infty} \|x_n\| = 1$, and $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$, by passing to subsequences, if necessary, we can assume that $1 \leq \|x_n\| \leq 1+\varepsilon_n$ for all $n \in \mathbb{N}$. Define $y_n = \frac{x_n}{1+\varepsilon_n}$ for all $n \in \mathbb{N}$. Then, since $\|y_n\| \leq 1$, we have

$$\left|\left|\left| \sum_{n=1}^{\infty} t_n y_n \right|\right|\right| \leq \sum_{n=1}^{\infty} |t_n| \leq \sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \quad \text{for all } (t_n)_n \in \ell^1 .$$

Also from the left hand side inequality of (2.8) and recalling ρ is a lattice norm, we have

$$\begin{aligned} \left|\left|\left| \sum_{n=1}^{\infty} t_n y_n \right|\right|\right| = \left|\left|\left| \sum_{n=1}^{\infty} t_n \frac{x_n}{1+\varepsilon_n} \right|\right|\right| &\geq \sum_{n=1}^{\infty} \frac{|t_n|}{1+\varepsilon_n} + \rho\left(\left(\frac{|t_n|}{1+\varepsilon_n} \right)_n \right) \\ &\geq \sum_{n=1}^{\infty} (1-\varepsilon_n)|t_n| + \rho\left(((1-\varepsilon_n)t_n)_n\right). \end{aligned}$$

□

Theorem 6. *Let $(X, \|.\|)$ be a Banach space. Then, X has property AAI2 if and only if X contains an ai copy of ℓ^1*

Proof. Let $(X, \|\cdot\|)$ be a Banach space and assume that X has property AAI2. So there exist a sequence $(x_n)_n$ in X satisfying $\lim_{n\to\infty} \|x_n\| = 1$, a constant $R \in [1,\infty)$, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ so that for any $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \leq \left|\left|\left| \sum_{n=1}^{\infty} t_n x_n \right|\right|\right| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \right] .$$

Then, there exist constants $0 < A \leq B < \infty$ such that for any $x = (\xi_n)_n \in c_0$, $A\rho(x) \leq \sum_{k=1}^{\infty} |\rho_k \xi_k| \leq B\rho(x)$. Let $M = R \left[1 + \sup_n \frac{|\rho_n|}{A}\right]$. Then,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n| .$$

So $(x_n)_n$ is an ai ℓ^1 sequence in X by Theorem 1.

Conversely assume that X contains an ai copy of ℓ^1. Then, by Theorem 1, there exist a sequence $(x_n)_n$ in X with $\lim_n \|x_n\| = 1$ and $M \in [1, \infty)$ such that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq M \sum_{n=1}^{\infty} |t_n| .$$

Now, fix a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and fix an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$. For example, we could consider any equivalent norm to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} \frac{|\xi_k|}{2^k}$ for any $x = (\xi_k)_k \in c_0$ but we may work on an arbitrary one. Then, letting $y_n = \left(1 + \frac{|\rho_n|}{A}\right) x_n$ for each $n \in \mathbb{N}$ and $R = \frac{BM}{A}$, we have

$$\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq R \left[\sum_{n=1}^{\infty} |t_n| + \rho\left((t_n)_n\right) \right] .$$

Hence, by the previous theorem, X has property AAI2. □

2.3. *Property AAI3; generalizing AAI1*

Using the ideas of the previous section, we may get a generalized version of property AAI1, which we call property AAI3. Indeed, one can easily obtain the following conclusions.

Corollary 1. *A Banach space $(X, \|\cdot\|)$ has property AAI3 if and only if there exist a sequence $(x_n)_n$ in X, a sequence $(\rho_n)_n \in \ell^1 \setminus c_{00}$ and an equivalent lattice norm ρ to $|||.|||$ given by $|||x||| = \sum_{k=1}^{\infty} |\rho_k \xi_k|$ for any $x = (\xi_k)_k \in c_0$ such that there exists $R \in [1, \infty)$ so that for any $(t_n)_n \in \ell^1$,*

$$\sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left[\rho\left((t_n)_n\right)\right]^2} \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq R \sqrt{\left(\sum_{n=1}^{\infty} |t_n|\right)^2 + \left[\rho\left((t_n)_n\right)\right]^2}$$

and $\lim_{n \to \infty} \|x_n\| = 1$.

Corollary 2. *Let $(X,\|.\|)$ be a Banach space. Then, X has property AAI3 if and only if X contains an ai copy of ℓ^1*

2.4. *Some Remarks and Examples for Alternatives to ai ℓ^1 Copy*

Example 1. We consider a renorming of ℓ^1 as follows. Consider $(\ell^1,\|.\|)$ with the norm defined by the following way: for every $x=(\xi_n)_n\in\ell^1$,

$$\|x\|=\sup_n \frac{2^n}{1+2^n}\sum_{k=n}^{\infty}|\xi_k|+\sum_{n=1}^{\infty}\frac{|\xi_n|}{2^n}+\frac{1}{3}\sum_{n=1}^{\infty}|\xi_n|\,.$$

Then, we can see that $(\ell^1,\|.\|)$ contains a sequence that satisfies both conditions of property AAI1 and an ai ℓ^1 sequence.

Proof. Fix an increasing sequence $(m_i)_i$ in $\mathbb{N}$ and define for each $k\in\mathbb{N}$, $M_k:=\sum_{i=1}^{k}m_i$ with $M_0:=0$ and $y_k=\frac{1}{1+m_k}\sum_{n=M_{k-1}+1}^{M_k}e_n$. Then, we get for each $k\in\mathbb{N}$,

$$\begin{aligned}\|y_k\| &\le \frac{1}{m_k+1}(1+\frac{5}{6}(M_k-M_{k-1}))\le\frac{1}{m_k+1}\left(1+\frac{5}{6}m_k\right)\\ &\le\frac{1}{m_k+1}(1+m_k)=1\end{aligned}$$

and thus, for all $(t_n)_n\in\ell^1$,

$$\left\|\sum_{n=1}^{\infty}t_n y_n\right\|\le\sum_{n=1}^{\infty}|t_n|\le\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}\,.$$

On the other hand, we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty}t_n y_n\right\| &= \left\| \begin{array}{l} t_1\frac{1}{m_1+1}\sum_{j=1}^{M_1}e_j+t_2\frac{1}{m_2+1}\sum_{j=M_1+1}^{M_2}e_j \\ +t_3\frac{1}{m_3+1}\sum_{j=M_2+1}^{M_3}e_j+\cdots+t_n\frac{1}{m_n+1}\sum_{j=M_{n-1}+1}^{M_n}e_j+\cdots \end{array}\right\|\\ &\ge\frac{2}{3}\sum_{n=1}^{\infty}\frac{|t_n|m_n}{m_n+1}+\sum_{n=1}^{\infty}\frac{m_n|t_n|}{2^n(m_n+1)}+\frac{1}{3}\sum_{n=1}^{\infty}\frac{m_n|t_n|}{m_n+1}\\ &=\sum_{n=1}^{\infty}|t_n|\frac{m_n}{m_n+1}\left(1+\frac{1}{2^n}\right)\\ &\ge\sqrt{\left(\sum_{n=1}^{\infty}|t_n|\frac{m_n}{m_n+1}\right)^2+\left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\frac{m_n}{m_n+1}\right)^2}\end{aligned}$$

Then, letting $m_n \leq 2^n$, $\varepsilon_n = \frac{1}{m_n+1}$ and $\delta_n = \frac{2^n - m_n}{2^n(m_n+1)}$ for each $n \in \mathbb{N}$, we obtain that for all $(t_n)_n \in \ell^1$,

$$\sum_{n=1}^{\infty} |t_n|(1-\delta_n) \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sum_{n=1}^{\infty} |t_n|$$

and

$$\sqrt{\left(\sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}(1-\varepsilon_n)\frac{|t_n|}{2^n}\right)^2} \leq \left\| \sum_{n=1}^{\infty} t_n y_n \right\| \leq \sqrt{\left(\sum_{n=1}^{\infty}|t_n|\right)^2 + \left(\sum_{n=1}^{\infty}\frac{|t_n|}{2^n}\right)^2}.$$

Thus, $(\ell^1, \|.\|)$ has property AAI1 and it contains ai copy of ℓ^1 such that both properties share a common sequence to exist. □

Remark 1. Dowling, Johnson, Lennard and Turett [2] constructed the following equivalent norm $|||\cdot|||^{\sim}$ on ℓ^1 and they showed that $\left(\ell^1, |||\cdot|||^{\sim}\right)$ does not contain an ai copy of ℓ^1 and later Lin [11] showed that ℓ^1 can be renormed to have fpp(ne) with a special version of the norm $|||\cdot|||^{\sim}$.

Equivalent norm $|||\cdot|||^{\sim}$ on ℓ^1 is given as follows: for $x = (\xi_k)_k \in \ell^1$, write $|||x|||^{\sim} := \sup_{k\in\mathbb{N}} \gamma_k \sum_{j=k}^{\infty} |\xi_j|$ where $\gamma_k \uparrow_k 1$, γ_k is strictly increasing.

Due to our result in previous section, having property AAI1 is equivalent for a Banach space to contain an ai copy of ℓ^1, we have a straight result that $\left(\ell^1, |||\cdot|||^{\sim}\right)$ does not satisfy AAI1 property.

Remark 2. Note that [5, Theorem 4.1 on page 87] says that if a Banach space contains an ai copy of c_0 then its isometric dual contains an ai copy of ℓ^1. Also note that the converse does not generally hold; that is, a dual Banach space may contain an ai copy of ℓ^1 but this does not generally imply its isometric predual contains an ai copy of c_0; e.g. as a predual, we can consider ℓ^1 with its canonical norm which does not contain an ai copy of c_0 but ℓ^∞ contains an isometric copy of ℓ^1 so it contains an ai copy of ℓ^1. Using the same theorem of Dowling, Lennard and Turett, we can say that if a Banach space fails AAI1 property, then its predual does not contain any ai copy of c_0.

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STATISTICAL CONVERGENCE OF CERTAIN CLASS OF POSITIVE LINEAR OPERATORS VIA THE POWER SERIES METHOD AND ITS APPLICATIONS TO KOROVKIN-TYPE THEOREM

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Abstract. We introduce the concept of P-statistical convergence of order α ($\mathcal{S}_P^\alpha$) for sequences of real numbers by using the power series density δ_P^α. We study several basic properties of this convergence method, including uniqueness of limits, linearity, and inclusion relations with ordinary convergence, statistical convergence of order α, and classical P-summability. We also provide examples to show that these inclusions are proper. As an application, we establish a Korovkin-type approximation theorem in the weighted space $C_\rho[0,\infty)$ with weight $\rho(x) = 1 + x^2$ by using the test functions $\{1, x, x^2\}$. Furthermore, we obtain a rate of convergence estimate by means of the weighted modulus of continuity. Finally, we apply the results to the Szász-Mirakjan-Kantorovich operators and obtain a P-statistical approximation rate of order α. A numerical example is included to illustrate the approximation behaviour of the proposed operators on an unbounded interval.

Keywords: power series method; statistical convergence of order α; weighted space; Szász-Mirakjan-Kantorovich operator; Korovkin-type theorem; weighted modulus of continuity

MSC 2020: 40G10, 40G15, 41A25, 41A36, 41A35

1. Introduction and motivation

The theory of summability is an important area in the study of divergent series, Fourier analysis, and approximation theory. Among the various summability methods, the power series method is one of the most general and widely used methods. It includes many classical methods such as Abel summability ($p_n = 1$), Borel summability ($p_n = 1/n!$), and the logarithmic method ($p_n = 1/(n+1)$) as special cases. More details about the power series method can be found in Boos [3] and Stieglitz and Tietz [11].

The idea of statistical convergence was introduced by Fast [4] and Steinhaus [10] by using natural density. Later, Ünver and Orhan [14] extended this concept to the

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power series method and introduced P-statistical convergence. A sequence is said to be P-statistically convergent to L if the set of indices for which the sequence differs from L by more than ε has P-density zero as the parameter t approaches the radius of convergence R. This method is stronger than ordinary P-summability and has important applications in approximation theory, especially in Korovkin-type approximation theorems studied by Atlihan et al. [1].

In approximation theory, the classical Korovkin theorem [7] was extended to unbounded intervals by Gadziev [5] through weighted continuous function spaces defined by

$$C_\rho[0,\infty) = \{f \in C[0,\infty)\colon\ |f(x)| \leqslant M_f(1+x^2)\},$$

with the weighted norm

$$\|f\|_\rho = \sup_{x\geqslant 0} \frac{|f(x)|}{1+x^2}.$$

In this space, the test functions $\{1, x, x^2\}$ are enough to prove Korovkin-type approximation results. The rate of approximation can also be studied by using the weighted modulus of continuity introduced by Holhoş [6].

Two important positive linear operators on the interval $[0,\infty)$ are the Szász-Mirakjan operators [8], [12]

$$S_n(f;x) = \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

and the Szász-Mirakjan-Kantorovich operators [2]

$$S_n^*(f;x) = n \sum_{k=0}^{\infty} \mathrm{e}^{-nx} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(u)\,\mathrm{d}u.$$

The Kantorovich form is very useful for approximating functions that are not smooth or only integrable, and it can be studied in both weighted spaces and L^p spaces [13].

Although these topics have been studied separately in many research works, the combined study of the power series method, statistical convergence of order α, and Szász-Mirakjan-Kantorovich operators in weighted spaces has not yet been investigated. The present paper aims to study these concepts together in a unified framework.

Motivated essentially by the above-mentioned investigation and study, we introduce the notion of P-statistical convergence of order α ($\mathcal{S}_P^\alpha$) and study its main properties. We also compare this new method with existing summability and convergence methods. Next, we establish a Korovkin-type approximation theorem in the weighted space $C_\rho[0,\infty)$ by using the test functions $\{1, x, x^2\}$. We then obtain

a quantitative estimate for the rate of convergence with the help of the weighted modulus of continuity introduced by Holhoş. Finally, we apply the developed theory to the Szász-Mirakjan-Kantorovich operators. We derive suitable moment estimates and prove a convergence rate of order $O(\omega_\rho(f; n^{-1/2}))$ in the $\mathcal{S}_P^\alpha$-sense. A numerical example on the interval $[0, 8]$ is also presented to show the smoothing and approximation behaviour of the proposed operators.

2. Preliminaries

In this paper, we use the notation $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Unless stated otherwise, all sequences are indexed by $\mathbb{N}_0$ and have real values. We write $C[0, \infty)$ for the set of all continuous real-valued functions defined on $[0, \infty)$.

Next, we introduce the following definitions.

Definition 2.1. Let $(p_n)_{n\in\mathbb{N}_0}$ be a sequence of nonnegative real numbers with $p_0 > 0$. Define

$$p(t) = \sum_{n=0}^{\infty} p_n t^n, \quad t \in [0, R),$$

where $R \in (0, \infty]$ is the radius of convergence of the series. Assume that

$$p(t) > 0 \quad \text{for all } t \in (0, R).$$

The power series method determined by (p_n) is denoted by $P = (p_n)$. For a sequence (x_n), its P-transform is defined by

$$P(x; t) = \frac{1}{p(t)} \sum_{n=0}^{\infty} p_n x_n t^n$$

whenever the series converges for $t \in (0, R)$. We say that the sequence (x_n) is P-summable to $L \in \mathbb{R}$ if

$$\lim_{t\to R^-} P(x; t) = L.$$

Remark 2.1. Different choices of (p_n) give different summability methods.

- ▷ If $p_n = 1$ for all n and $R = 1$, then we obtain the Abel method.
- ▷ If $p_n = 1/n!$ for all n and $R = \infty$, then we obtain the Borel method.
- ▷ If $p_n = 1/(n + 1)$ for all n and $R = 1$, then we obtain the logarithmic method.

Each of these methods is regular, meaning that they preserve ordinary limits.

Definition 2.2. Let $P = (p_n)$ be a power series method as in Definition 2.1, and let $\alpha \in (0, 1]$. For a set $A \subset \mathbb{N}_0$, define

$$\delta_P^\alpha(A) = \lim_{t \to R^-} \frac{1}{[p(t)]^\alpha} \sum_{n \in A} p_n t^n$$

whenever the limit exists. The quantity $\delta_P^\alpha(A)$ is called the P-density of order α of the set A.

Remark 2.2. When $\alpha = 1$, the quantity δ_P^α is exactly the same as the P-density introduced by Ünver and Orhan [14]. If $\alpha < 1$, then $[p(t)]^\alpha$ grows more slowly than $p(t)$ as $t \to R^-$. Because of this, the condition for a set to have δ_P^α-density zero becomes stronger. In other words, a set A is δ_P^α-null only when

$$\sum_{n \in A} p_n t^n$$

approaches zero faster compared to the normalizing factor $[p(t)]^\alpha$.

Remark 2.3. The P-statistical convergence of order α introduced here extends the existing statistical convergence of order α [9] by replacing the natural-density normalization n^α with the power-series normalization $[p(t)]^\alpha$. In fact, statistical convergence of order α is characterized by

$$\lim_{n \to \infty} \frac{1}{n^\alpha} |\{k \leqslant n \colon |x_k - L| \geqslant \varepsilon\}| = 0,$$

whereas P-statistical convergence of order α is defined by

$$\lim_{t \to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{k \geqslant 0 \\ |x_k - L| \geqslant \varepsilon}} p_k t^k = 0.$$

Therefore, P-statistical convergence of order α can be viewed as a power-series generalization of statistical convergence of order α.

Definition 2.3. Let $(x_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers. We say that (x_n) is P-statistically convergent of order α to $L \in \mathbb{R}$, and write

$$\mathcal{S}_{P^-}^\alpha \lim_{n \to \infty} x_n = L$$

if for every $\varepsilon > 0$,

$$\lim_{t \to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n \in \mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n = 0. \tag{2.1}$$

This means that the set of indices for which x_n is not close to L has P-density of order α equal to zero. The collection of all sequences that satisfy this condition is denoted by $\mathcal{S}_P^\alpha$.

We now recall the weighted continuous function spaces.

Following Gadziev [5], let

$$\rho(x) = 1 + x^2$$

and define

$$C_\rho[0,\infty) = \left\{ f \in C[0,\infty) \colon \sup_{x \geqslant 0} \frac{|f(x)|}{\rho(x)} < \infty \right\}.$$

Thus, $C_\rho[0,\infty)$ is the set of all continuous functions on $[0,\infty)$ whose growth is controlled by $\rho(x) = 1 + x^2$. For $f \in C_\rho[0,\infty)$, the weighted norm is defined by

$$\|f\|_\rho = \sup_{x \geqslant 0} \frac{|f(x)|}{\rho(x)}.$$

We also define the subspace

$$C_\rho^k[0,\infty) = \left\{ f \in C_\rho[0,\infty) \colon \lim_{x \to \infty} \frac{f(x)}{\rho(x)} \text{ exists and is finite} \right\}.$$

This means that $C_\rho^k[0,\infty)$ contains all functions in $C_\rho[0,\infty)$ for which the ratio

$$\frac{f(x)}{\rho(x)}$$

approaches a finite limit as $x \to \infty$. Moreover, $C_\rho^k[0,\infty)$ is a closed subspace of the normed space

$$(C_\rho[0,\infty), \|\cdot\|_\rho).$$

Next, we recall the following theorem.

Theorem 2.1 (see [5]). *Let (L_n) be a sequence of positive linear operators from $C_\rho^k[0,\infty)$ into $C_\rho[0,\infty)$. Assume that*

$$\lim_{n \to \infty} \|L_n e_j - e_j\|_\rho = 0, \quad j = 0, 1, 2,$$

where

$$e_j(x) = x^j.$$

Then for every function $f \in C_\rho^k[0,\infty)$,

$$\lim_{n \to \infty} \|L_n f - f\|_\rho = 0.$$

We now recall the weighted modulus of continuity.

The classical modulus of continuity is not suitable on $[0,\infty)$. Therefore, we use the weighted modulus of continuity introduced by Holhoş [6].

Let $f \in C_\rho[0,\infty)$ and let $\delta > 0$. The weighted modulus of continuity introduced by Holhoş [6] is defined by

$$\omega_\rho(f;\delta) := \sup_{\substack{x,y\geqslant 0\\ |\varphi(x)-\varphi(y)|\leqslant\delta}} \frac{|f(x)-f(y)|}{\rho(x)+\rho(y)},$$

where

$$\varphi(x) = \frac{x}{1+x}.$$

The function

$$\varphi(x) = \frac{x}{1+x}$$

maps the interval $[0,\infty)$ into $[0,1)$. This helps control the behavior of the modulus near infinity.

The following properties of $\omega_\rho(f;\delta)$ are given in [6]:

- $\omega_\rho(f;\delta)$ is nondecreasing in δ;
- if $f \in C_\rho^k[0,\infty)$, then
$$\lim_{\delta\to 0^+} \omega_\rho(f;\delta) = 0;$$
- for every $\lambda > 0$,
$$\omega_\rho(f;\lambda\delta) \leqslant 2(1+\lambda)\omega_\rho(f;\delta).$$

Next, we introduce the Szász-Mirakjan operators and the Szász-Mirakjan-Kantorovich operators, which will be used in this work.

For $n \in \mathbb{N}$ and $f \in C_\rho[0,\infty)$, the Szász-Mirakjan operator is defined by

$$S_n(f;x) = \mathrm{e}^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \quad x \geqslant 0. \tag{2.2}$$

If f is locally integrable, then the Szász-Mirakjan-Kantorovich operator is defined by

$$S_n^*(f;x) = n \sum_{k=0}^{\infty} \mathrm{e}^{-nx} \frac{(nx)^k}{k!} \int_{k/n}^{(k+1)/n} f(u)\,\mathrm{d}u, \quad x \geqslant 0. \tag{2.3}$$

Both operators are positive and linear. For the test functions

$$e_j(x) = x^j,$$

their moments are given as follows.

For the Szász-Mirakjan operators,

$$S_n(e_0;x)=1,\quad S_n(e_1;x)=x,\quad S_n(e_2;x)=x^2+\frac{x}{n}. \tag{2.4}$$

For the Szász-Mirakjan-Kantorovich operators,

$$S_n^*(e_0;x)=1,\quad S_n^*(e_1;x)=x+\frac{1}{2n},\quad S_n^*(e_2;x)=x^2+\frac{2x}{n}+\frac{1}{3n^2}. \tag{2.5}$$

3. P-statistical convergence of order α

In this section, we introduce and study the basic and fundamental properties of the P-statistical convergence of order α.

Theorem 3.1. *If a sequence (x_n) is P-statistically convergent of order α, then its limit is unique. That is, if*

$$\mathcal{S}_{P^-}^{\alpha}\lim_{n\to\infty}x_n=L$$

and

$$\mathcal{S}_{P^-}^{\alpha}\lim_{n\to\infty}x_n=M,$$

then

$$L=M.$$

P r o o f. Assume that

$$\mathcal{S}_{P^-}^{\alpha}\lim_{n\to\infty}x_n=L$$

and

$$\mathcal{S}_{P^-}^{\alpha}\lim_{n\to\infty}x_n=M.$$

We will show that $L=M$.

Suppose, on the contrary, that $L\neq M$. Then

$$|L-M|>0.$$

Choose

$$\varepsilon=\frac{|L-M|}{2}>0.$$

Now, for every $n\in\mathbb{N}_0$, it is impossible that both

$$|x_n-L|<\varepsilon$$

and

$$|x_n - M| < \varepsilon$$

hold at the same time. Indeed, if both inequalities were true, then by the triangle inequality,

$$|L - M| \leqslant |L - x_n| + |x_n - M| < \varepsilon + \varepsilon = |L - M|,$$

which is impossible.

Hence, for every $n \in \mathbb{N}_0$, at least one of the inequalities

$$|x_n - L| \geqslant \varepsilon \quad \text{or} \quad |x_n - M| \geqslant \varepsilon$$

must hold.

Therefore,

$$\sum_{n=0}^{\infty} p_n t^n \leqslant \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L|\geqslant\varepsilon}} p_n t^n + \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - M|\geqslant\varepsilon}} p_n t^n.$$

Dividing by $[p(t)]^\alpha$, we obtain

$$\frac{p(t)}{[p(t)]^\alpha} \leqslant \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L|\geqslant\varepsilon}} p_n t^n + \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - M|\geqslant\varepsilon}} p_n t^n.$$

Since (x_n) is P-statistically convergent of order α to both L and M, taking the limit as $t \to R^-$ gives

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L|\geqslant\varepsilon}} p_n t^n = 0$$

and

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - M|\geqslant\varepsilon}} p_n t^n = 0.$$

Hence,

$$\lim_{t\to R^-} \frac{p(t)}{[p(t)]^\alpha} = 0.$$

But

$$\frac{p(t)}{[p(t)]^\alpha} = [p(t)]^{1-\alpha}.$$

Since $\alpha \in (0, 1]$ and $p(t) > 0$, the above expression cannot converge to 0. This is a contradiction.

Therefore, our assumption $L \neq M$ is false. Hence,

$$L = M.$$

Thus the P-statistical limit of order α is unique. □

Theorem 3.2 (Monotonicity with respect to α)**.** *Let $0 < \alpha \leqslant \beta \leqslant 1$. If a sequence is P-statistically convergent of order α, then it is also P-statistically convergent of order β.*

In other words,

$$\mathcal{S}_P^{\alpha} \subset \mathcal{S}_P^{\beta}.$$

P r o o f. Let

$$0 < \alpha \leqslant \beta \leqslant 1,$$

and suppose that

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} x_n = L.$$

Then by Definition 2.3, for every $\varepsilon > 0$,

$$\lim_{t\to R^-} \frac{1}{[p(t)]^{\alpha}} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n = 0. \tag{3.1}$$

We must show that

$$\mathcal{S}_{P^-}^{\beta} \lim_{n\to\infty} x_n = L.$$

Since $\alpha \leqslant \beta$ and $p(t) > 0$, we have

$$[p(t)]^{\beta} \geqslant [p(t)]^{\alpha}.$$

Therefore,

$$\frac{1}{[p(t)]^{\beta}} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n \leqslant \frac{1}{[p(t)]^{\alpha}} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n.$$

Taking the limit as $t \to R^-$ and using (3.1), we obtain

$$\lim_{t\to R^-} \frac{1}{[p(t)]^{\beta}} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n = 0.$$

Hence (x_n) is P-statistically convergent of order β to L. Therefore,

$$\mathcal{S}_P^{\alpha} \subset \mathcal{S}_P^{\beta}.$$

□

Theorem 3.3 (Stability under bounded $\mathcal{S}_P^\alpha$-null perturbations)**.** *Suppose that*

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} x_n = L$$

and

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} y_n = 0,$$

where the sequence (y_n) is bounded.

Then

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} (x_n + y_n) = L.$$

In other words, adding a bounded sequence that is P-statistically convergent of order α to zero does not change the P-statistical limit of order α.

P r o o f. Assume that

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} x_n = L$$

and

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} y_n = 0,$$

where (y_n) is bounded.

We will show that

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} (x_n + y_n) = L.$$

Let $\varepsilon > 0$ be given. Since

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} y_n = 0,$$

we have

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |y_n|\geqslant\varepsilon/2}} p_n t^n = 0.$$

Also, since

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} x_n = L,$$

we have

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon/2}} p_n t^n = 0.$$

Now suppose that

$$|(x_n + y_n) - L| \geqslant \varepsilon.$$

By the triangle inequality,

$$|(x_n + y_n) - L| \leqslant |x_n - L| + |y_n|.$$

Hence, if

$$|(x_n + y_n) - L| \geqslant \varepsilon,$$

then at least one of the following inequalities must hold:

$$|x_n - L| \geqslant \frac{\varepsilon}{2}$$

or

$$|y_n| \geqslant \frac{\varepsilon}{2}.$$

Therefore,

$$\{n \in \mathbb{N}_0 \colon\ |(x_n + y_n) - L| \geqslant \varepsilon\}$$

is contained in

$$\left\{n \in \mathbb{N}_0 \colon\ |x_n - L| \geqslant \frac{\varepsilon}{2}\right\} \cup \left\{n \in \mathbb{N}_0 \colon\ |y_n| \geqslant \frac{\varepsilon}{2}\right\}.$$

Using this inclusion, we obtain

$$\sum_{\substack{n\in\mathbb{N}_0\\ |(x_n+y_n)-L|\geqslant\varepsilon}} p_n t^n \leqslant \sum_{\substack{n\in\mathbb{N}_0\\ |x_n-L|\geqslant\varepsilon/2}} p_n t^n + \sum_{\substack{n\in\mathbb{N}_0\\ |y_n|\geqslant\varepsilon/2}} p_n t^n.$$

Dividing by $[p(t)]^\alpha$, we get

$$\frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0\\ |(x_n+y_n)-L|\geqslant\varepsilon}} p_n t^n \leqslant \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0\\ |x_n-L|\geqslant\varepsilon/2}} p_n t^n + \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0\\ |y_n|\geqslant\varepsilon/2}} p_n t^n.$$

Taking the limit as $t \to R^-$, both terms on the right-hand side converge to 0. Hence,

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0\\ |(x_n+y_n)-L|\geqslant\varepsilon}} p_n t^n = 0.$$

Therefore,

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} (x_n + y_n) = L.$$

□

4. Inclusion relations and counterexamples

In this section, we study the relationship between $\mathcal{S}_P^\alpha$ and some well-known types of convergence. We write c for the space of all ordinarily convergent sequences, c_P for the space of all P-summable sequences, and c_P^α for the space $\mathcal{S}_P^\alpha$ of all P-statistically convergent sequences of order α.

Throughout this section, we assume that the power series method P is regular.

Theorem 4.1. *Assume that P is regular. Then the following inclusion relations hold.*

(i) *Every ordinarily convergent sequence is P-statistically convergent of order α. That is,*

$$c \subset \mathcal{S}_P^\alpha, \quad \alpha \in (0,1].$$

(ii) *If $0 < \alpha \leqslant 1$, then every sequence that is P-statistically convergent of order α is also P-statistically convergent of order 1. In other words,*

$$\mathcal{S}_P^\alpha \subset \mathcal{S}_P^1.$$

(iii) *Every bounded sequence that is P-statistically convergent of order α is also P-summable. Thus,*

$$\mathcal{S}_P^\alpha \cap \ell^\infty \subset c_P, \quad \alpha \in (0,1].$$

P r o o f. We prove each part separately.

(i) Proof that $c \subset \mathcal{S}_P^\alpha$.

Assume that $(x_n) \in c$. Then there exists $L \in \mathbb{R}$ such that

$$\lim_{n\to\infty} x_n = L.$$

Let $\varepsilon > 0$ be given. Since $x_n \to L$, there exists $N \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \quad \text{for all } n \geqslant N.$$

Hence,

$$A_\varepsilon := \{n \in \mathbb{N}_0 \colon\ |x_n - L| \geqslant \varepsilon\}$$

is a finite set.

Therefore,

$$\sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n = \sum_{n\in A_\varepsilon} p_n t^n.$$

Since A_ε is finite, the above sum is bounded, while

$$[p(t)]^\alpha \to \infty \quad \text{as } t \to R^-.$$

Thus,

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n = 0.$$

Hence,

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} x_n = L.$$

Therefore,

$$c \subset \mathcal{S}^\alpha_P.$$

(ii) Proof that $\mathcal{S}^\alpha_P \subset \mathcal{S}^1_P$.
Let

$$0 < \alpha \leqslant 1,$$

and suppose that

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} x_n = L.$$

Then for every $\varepsilon > 0$,

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n = 0.$$

Since $\alpha \leqslant 1$, we have

$$[p(t)]^\alpha \leqslant p(t).$$

Therefore,

$$\frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n \leqslant \frac{1}{[p(t)]^\alpha} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n.$$

Taking the limit as $t \to R^-$, we obtain

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n-L|\geqslant\varepsilon}} p_n t^n = 0.$$

Hence,

$$\mathcal{S}^1_{P^-} \lim_{n\to\infty} x_n = L.$$

Therefore,

$$\mathcal{S}^\alpha_P \subset \mathcal{S}^1_P.$$

(iii) Proof that $\mathcal{S}_P^\alpha \cap \ell^\infty \subset c_P$.
Suppose that (x_n) is bounded and

$$\mathcal{S}_{P^-}^\alpha \lim_{n\to\infty} x_n = L.$$

Since (x_n) is bounded, there exists a constant $M > 0$ such that

$$|x_n - L| \leqslant M \quad \text{for all } n.$$

We write

$$P(x;t) - L = \frac{1}{p(t)} \sum_{n=0}^{\infty} p_n (x_n - L) t^n.$$

Let $\varepsilon > 0$ be fixed. Split the sum into two parts:

$$|P(x;t) - L| \leqslant \frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| < \varepsilon}} p_n |x_n - L| t^n + \frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n |x_n - L| t^n.$$

For the first sum,

$$\frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| < \varepsilon}} p_n |x_n - L| t^n \leqslant \varepsilon.$$

For the second sum, using $|x_n - L| \leqslant M$, we get

$$\frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n |x_n - L| t^n \leqslant M \frac{1}{p(t)} \sum_{\substack{n\in\mathbb{N}_0 \\ |x_n - L| \geqslant \varepsilon}} p_n t^n.$$

Since

$$\mathcal{S}_P^\alpha \subset \mathcal{S}_P^1,$$

the last term converges to 0 as $t \to R^-$.
Therefore,

$$\limsup_{t\to R^-} |P(x;t) - L| \leqslant \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, we conclude that

$$\lim_{t\to R^-} P(x;t) = L.$$

Hence $(x_n) \in c_P$, and therefore

$$\mathcal{S}_P^\alpha \cap \ell^\infty \subset c_P.$$

□

Next, we present the following examples of Theorem 4.1.

Example 4.1. (i) Consider the sequence

$$x_n = \frac{1}{n+1}, \quad n \in \mathbb{N}_0.$$

Clearly,

$$\lim_{n\to\infty} x_n = 0.$$

Hence $(x_n) \in c$. By part (i) of Theorem 4.1, the sequence is also P-statistically convergent of order α to 0, that is,

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} x_n = 0.$$

This example shows that every ordinarily convergent sequence belongs to $\mathcal{S}_P^{\alpha}$.
(ii) Let

$$x_n = \begin{cases} 1 & \text{if } n = m^2 \quad \text{for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The set of perfect squares is sparse inside $\mathbb{N}_0$. Hence the sequence is P-statistically convergent of order α to 0 for every $\alpha \in (0, 1]$:

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} x_n = 0.$$

By part (ii) of Theorem 4.1, it follows that

$$\mathcal{S}_{P^-}^{1} \lim_{n\to\infty} x_n = 0.$$

This example illustrates that

$$\mathcal{S}_P^{\alpha} \subset \mathcal{S}_P^{1}.$$

(iii) Consider the bounded sequence

$$x_n = \begin{cases} 1 & \text{if } n = m^2 \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $|x_n| \leqslant 1$ for all n, the sequence belongs to ℓ^{∞}.
Also, as in the previous example,

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} x_n = 0.$$

Therefore, by part (iii) of Theorem 4.1, the sequence is P-summable to 0:

$$\lim_{t\to R^-} P(x;t) = 0.$$

Hence,

$$(x_n) \in c_P.$$

This example shows that every bounded sequence in $\mathcal{S}_P^{\alpha}$ is also P-summable.

Combining the above, we obtain the strict hierarchy

$$c \subsetneq \mathcal{S}_P^{\alpha} \subsetneq \mathcal{S}_P^{1} \subsetneq c_P \text{ (on } \ell^{\infty})$$

for every $\alpha \in (0,1)$ and every regular power series method P.

5. A Korovkin-type theorem in $C_\rho[0,\infty)$ under $\mathcal{S}_P^{\alpha}$

We now adapt Gadziev's [5] weighted Korovkin theorem to the $\mathcal{S}_P^{\alpha}$ framework.

Theorem 5.1. *Let $(L_n)_{n\in\mathbb{N}_0}$ be a sequence of positive linear operators from $C_\rho^k[0,\infty)$ into $C_\rho[0,\infty)$, and let $\alpha \in (0,1]$. Then the following statements are equivalent.*

(i) *For every function $f \in C_\rho^k[0,\infty)$,*

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

That is, the operators L_n approximate every function in $C_\rho^k[0,\infty)$ in the weighted norm $\|\cdot\|_\rho$ with respect to P-statistical convergence of order α.

(ii) *For the test functions*

$$e_0(x) = 1, \quad e_1(x) = x, \quad e_2(x) = x^2,$$

we have

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n e_j - e_j\|_\rho = 0, \quad j = 0,1,2.$$

That is, it is enough to check the convergence only for the three test functions

$$1, \quad x, \quad x^2.$$

Proof. We prove that (i) and (ii) are equivalent.

For (i) $\Rightarrow$ (ii).

Assume that

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0 \quad \text{for every } f \in C_\rho^k[0,\infty).$$

Since the test functions

$$e_0(x) = 1, \quad e_1(x) = x, \quad e_2(x) = x^2$$

belong to $C_\rho^k[0,\infty)$, we can choose $f = e_j$ for $j = 0, 1, 2$. Then

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n e_j - e_j\|_\rho = 0, \quad j = 0, 1, 2.$$

Hence (ii) holds.

For (ii) $\Rightarrow$ (i).

Assume that

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n e_j - e_j\|_\rho = 0, \quad j = 0, 1, 2.$$

We must show that for every $f \in C_\rho^k[0,\infty)$,

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Let $\varepsilon > 0$ be given. Define the sets

$$A_j(\varepsilon) = \left\{ n \in \mathbb{N}_0 \colon \|L_n e_j - e_j\|_\rho \geqslant \frac{\varepsilon}{3} \right\}, \quad j = 0, 1, 2.$$

Since each test function converges P-statistically of order α, we have

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{n\in A_j(\varepsilon)} p_n t^n = 0, \quad j = 0, 1, 2.$$

Now let

$$A(\varepsilon) = \{n \in \mathbb{N}_0 \colon \|L_n f - f\|_\rho \geqslant \varepsilon\}.$$

By the classical weighted Korovkin theorem (Theorem 2.1), convergence of the three test functions implies convergence for every function $f \in C_\rho^k[0,\infty)$. Hence there exists a constant $C > 0$ such that

$$\|L_n f - f\|_\rho \leqslant C \sum_{j=0}^{2} \|L_n e_j - e_j\|_\rho.$$

Therefore, if

$$\|L_n f - f\|_\rho \geqslant \varepsilon,$$

then at least one of the inequalities

$$\|L_n e_j - e_j\|_\rho \geqslant \frac{\varepsilon}{3C}, \quad j = 0, 1, 2,$$

must hold.

Hence,

$$A(\varepsilon) \subset A_0\left(\frac{\varepsilon}{3C}\right) \cup A_1\left(\frac{\varepsilon}{3C}\right) \cup A_2\left(\frac{\varepsilon}{3C}\right).$$

Using this inclusion, we obtain

$$\sum_{n\in A(\varepsilon)} p_n t^n \leqslant \sum_{j=0}^{2} \sum_{n\in A_j(\varepsilon/(3C))} p_n t^n.$$

Dividing by $[p(t)]^\alpha$, we get

$$\frac{1}{[p(t)]^\alpha} \sum_{n\in A(\varepsilon)} p_n t^n \leqslant \sum_{j=0}^{2} \frac{1}{[p(t)]^\alpha} \sum_{n\in A_j(\varepsilon/(3C))} p_n t^n.$$

Taking the limit as $t \to R^-$, each term on the right-hand side converges to 0. Therefore,

$$\lim_{t\to R^-} \frac{1}{[p(t)]^\alpha} \sum_{n\in A(\varepsilon)} p_n t^n = 0.$$

This shows that

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Hence (i) holds. Therefore, (i) and (ii) are equivalent. □

Remark 5.1. The constants appearing in the previous result can be written more explicitly.

Let $f \in C^k_\rho[0,\infty)$ and suppose that

$$|f(x)| \leqslant M\rho(x)$$

for a constant $M > 0$. Using the estimate

$$|f(y) - f(x)| \leqslant 2\left(1 + \frac{(y-x)^2}{\delta^2}\right)\omega_\rho(f;\delta)(\rho(x) + \rho(y)),$$

we obtain the following bounds:

$$K_0 = M + 2\omega_\rho(f;\delta),$$
$$K_1 \leqslant \frac{2(1+b)}{\delta^2}\omega_\rho(f;\delta),$$

and

$$K_2 \leqslant \frac{2}{\delta^2}\omega_\rho(f;\delta),$$

for any finite interval

$$[0, b].$$

The behavior outside this interval, that is, near infinity, is controlled by the condition

$$f \in C_\rho^k[0, \infty).$$

6. Rate of convergence via the weighted modulus

Theorem 6.1. *Let (L_n) be a sequence of positive linear operators on $C_\rho[0, \infty)$ such that*

$$L_n e_0 = e_0, \quad e_0(x) = 1.$$

Then for every $f \in C_\rho[0, \infty)$, every $n \in \mathbb{N}$, and every $\delta > 0$, we have

$$\|L_n f - f\|_\rho \leqslant (2 + \|L_n \rho - \rho\|_\rho)\omega_\rho\Big(f; \sqrt{\|L_n \phi^2 - \phi^2\|_\rho + \Lambda_n}\Big),$$

where

$$\phi(x) = \frac{x}{1 + x},$$

and

$$\Lambda_n := \|L_n((\phi(\cdot) - \phi(x))^2; x)\|_\rho.$$

Equivalently, since $L_n e_0 = e_0$,

$$\Lambda_n = \|L_n(\phi^2; x) - 2\phi(x)L_n(\phi; x) + \phi^2(x)\|_\rho.$$

Here, $\omega_\rho(f; \cdot)$ denotes the weighted modulus of continuity of f.

P r o o f. Let $f \in C_\rho[0, \infty)$. Since L_n is positive and linear and

$$L_n e_0 = e_0,$$

we estimate the quantity

$$|L_n(f; x) - f(x)|.$$

For any $x, y \geqslant 0$, by the definition of the weighted modulus of continuity,

$$|f(y) - f(x)| \leqslant (\rho(x) + \rho(y))\omega_\rho(f; |\varphi(y) - \varphi(x)|),$$

where

$$\varphi(x) = \frac{x}{1 + x}.$$

Using the property

$$\omega_\rho(f;\lambda\delta) \leqslant 2(1+\lambda)\omega_\rho(f;\delta)$$

with

$$\lambda = \frac{|\varphi(y)-\varphi(x)|}{\delta},$$

we obtain

$$|f(y)-f(x)| \leqslant 2\left(1+\frac{|\varphi(y)-\varphi(x)|}{\delta}\right)(\rho(x)+\rho(y))\omega_\rho(f;\delta).$$

Now apply the operator L_n with respect to the variable y. Since L_n is positive and linear,

$$\begin{aligned}|L_n(f;x)-f(x)| &\leqslant L_n(|f(y)-f(x)|;x)\\ &\leqslant 2\omega_\rho(f;\delta)L_n\left(\left(1+\frac{|\varphi(y)-\varphi(x)|}{\delta}\right)(\rho(x)+\rho(y));x\right).\end{aligned}$$

Because $\rho(x)$ does not depend on y, we have

$$L_n(\rho(x);x) = \rho(x)L_n(e_0;x) = \rho(x).$$

Hence,

$$\begin{aligned}&|L_n(f;x)-f(x)|\\ &\quad\leqslant 2\omega_\rho(f;\delta)\left[\rho(x)+L_n(\rho;y;x)+\frac{1}{\delta}L_n(|\varphi(y)-\varphi(x)|(\rho(x)+\rho(y));x)\right].\end{aligned}$$

Using the Cauchy-Schwarz inequality

$$L_n(|ab|;x) \leqslant \sqrt{L_n(a^2;x)}\sqrt{L_n(b^2;x)},$$

we obtain

$$\begin{aligned}&L_n(|\varphi(y)-\varphi(x)|(\rho(x)+\rho(y));x)\\ &\quad\leqslant \sqrt{L_n((\varphi(y)-\varphi(x))^2;x)}\sqrt{L_n((\rho(x)+\rho(y))^2;x)}.\end{aligned}$$

Now define

$$\Lambda_n := \|L_n((\phi(\cdot)-\phi(x))^2;x)\|_\rho.$$

Also,

$$\|L_n\rho-\rho\|_\rho = \sup_{x\geqslant 0}\frac{|L_n\rho(x)-\rho(x)|}{\rho(x)}.$$

Combining the above estimates and dividing by $\rho(x)$, we get

$$\|L_n f - f\|_\rho \leqslant (2 + \|L_n\rho - \rho\|_\rho)\omega_\rho(f;\delta)\left(1 + \frac{\sqrt{\Lambda_n}}{\delta}\right).$$

Finally, choose

$$\delta = \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}.$$

Substituting this value of δ gives

$$\|L_n f - f\|_\rho \leqslant (2 + \|L_n\rho - \rho\|_\rho)\omega_\rho\Big(f; \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}\Big).$$

This completes the proof. □

Next, we present the following corollary as a consequence of Theorem 6.1.

Corollary 6.1. *Suppose that*

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \|L_n\varphi^2 - \varphi^2\|_\rho = 0$$

and

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \Lambda_n = 0,$$

where

$$\varphi(x) = \frac{x}{1+x}.$$

Then for every function $f \in C^k_\rho[0,\infty)$,

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Moreover, the speed of approximation is controlled by

$$\omega_\rho(f; \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}).$$

That is, the approximation becomes better as

$$\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n$$

approaches zero.

P r o o f. By Theorem 6.1, for every $f \in C_\rho[0,\infty)$,

$$\|L_n f - f\|_\rho \leqslant (2 + \|L_n\rho - \rho\|_\rho)\omega_\rho\Big(f; \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}\Big).$$

Assume that

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \|L_n\varphi^2 - \varphi^2\|_\rho = 0$$

and

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \Lambda_n = 0.$$

Then

$$\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n$$

is P-statistically convergent of order α to 0. Hence,

$$\sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n} \to 0$$

in the sense of P-statistical convergence of order α.

Since $f \in C^k_\rho[0,\infty)$, the weighted modulus of continuity satisfies

$$\lim_{\delta\to 0^+} \omega_\rho(f;\delta) = 0.$$

Therefore,

$$\omega_\rho\Big(f; \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}\Big)$$

also converges to 0 in the sense of P-statistical convergence of order α.

Moreover, the quantity

$$2 + \|L_n\rho - \rho\|_\rho$$

is nonnegative. Hence the right-hand side of the inequality converges to 0 in the sense of P-statistical convergence of order α.

Consequently,

$$\mathcal{S}^\alpha_{P^-} \lim_{n\to\infty} \|L_n f - f\|_\rho = 0.$$

Thus, for every $f \in C^k_\rho[0,\infty)$, the operators L_n approximate f in the weighted norm, and the rate of approximation is controlled by

$$\omega_\rho\Big(f; \sqrt{\|L_n\varphi^2 - \varphi^2\|_\rho + \Lambda_n}\Big).$$

□

7. Applications

We now study the Szász-Mirakjan-Kantorovich operators S_n^* defined in (2.3). The moment formulas given in (2.5) will be used in all the estimates that follow.

Lemma 7.1. *Let S_n^* be the Szász-Mirakjan-Kantorovich operators defined by (2.3), and let*

$$\rho(x) = 1 + x^2.$$

Then the following estimates hold:

$$\|S_n^* e_0 - e_0\|_\rho = 0, \tag{7.1}$$

$$\|S_n^* e_1 - e_1\|_\rho \leqslant \frac{1}{2n}, \tag{7.2}$$

$$\|S_n^* e_2 - e_2\|_\rho \leqslant \frac{1}{n} + \frac{1}{3n^2}. \tag{7.3}$$

That is, the operators S_n^* approximate the test functions

$$1, \quad x, \quad x^2$$

with the above error bounds in the weighted norm $\|\cdot\|_\rho$.

Moreover, the second central moment of S_n^* is given by

$$S_n^*((u-x)^2; x) = \frac{x}{n} + \frac{1}{3n^2}. \tag{7.4}$$

P r o o f. Using the moment formulas in (2.5), we prove each part separately.

From (2.5),

$$S_n^*(e_0; x) = 1 = e_0(x).$$

Hence,

$$S_n^* e_0 - e_0 = 0.$$

Therefore,

$$\|S_n^* e_0 - e_0\|_\rho = 0.$$

Again, from (2.5),

$$S_n^*(e_1; x) = x + \frac{1}{2n}.$$

Thus,

$$S_n^*(e_1; x) - e_1(x) = \frac{1}{2n}.$$

Using the weighted norm

$$\|f\|_\rho = \sup_{x \geqslant 0} \frac{|f(x)|}{1+x^2},$$

we obtain

$$\|S_n^* e_1 - e_1\|_\rho = \sup_{x \geqslant 0} \frac{1/(2n)}{1+x^2}.$$

Since

$$1 + x^2 \geqslant 1,$$

we get

$$\frac{1}{1+x^2} \leqslant 1.$$

Hence,

$$\|S_n^* e_1 - e_1\|_\rho \leqslant \frac{1}{2n}.$$

Next, from (2.5),

$$S_n^*(e_2; x) = x^2 + \frac{2x}{n} + \frac{1}{3n^2}.$$

Therefore,

$$S_n^*(e_2; x) - e_2(x) = \frac{2x}{n} + \frac{1}{3n^2}.$$

Hence,

$$\|S_n^* e_2 - e_2\|_\rho = \sup_{x \geqslant 0} \frac{2x/n + 1/(3n^2)}{1+x^2}.$$

Now,

$$\frac{2x}{1+x^2} \leqslant 1 \quad \text{for all } x \geqslant 0.$$

Thus,

$$\frac{2x/n}{1+x^2} \leqslant \frac{1}{n}.$$

Also,

$$\frac{1/(3n^2)}{1+x^2} \leqslant \frac{1}{3n^2}.$$

Combining these estimates, we obtain

$$\|S_n^* e_2 - e_2\|_\rho \leqslant \frac{1}{n} + \frac{1}{3n^2}.$$

We use the identity

$$(u-x)^2 = u^2 - 2xu + x^2.$$

Applying the operator S_n^*, we get

$$S_n^*((u-x)^2; x) = S_n^*(u^2; x) - 2x S_n^*(u; x) + x^2 S_n^*(1; x).$$

Using the moment formulas,

$$S_n^*(u^2;x) = x^2 + \frac{2x}{n} + \frac{1}{3n^2},$$

$$S_n^*(u;x) = x + \frac{1}{2n},$$

and

$$S_n^*(1;x) = 1.$$

Substituting these values, we obtain

$$\begin{aligned} S_n^*((u-x)^2;x) &= \left(x^2 + \frac{2x}{n} + \frac{1}{3n^2}\right) - 2x\left(x + \frac{1}{2n}\right) + x^2 \\ &= x^2 + \frac{2x}{n} + \frac{1}{3n^2} - 2x^2 - \frac{x}{n} + x^2. \end{aligned}$$

After simplification,

$$S_n^*((u-x)^2;x) = \frac{x}{n} + \frac{1}{3n^2}.$$

This completes the proof. □

Theorem 7.1. *Let $\alpha \in (0,1]$, and let P be a regular power series method. Assume that*

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \frac{1}{n} = 0.$$

Then for every function $f \in C_\rho^k[0,\infty)$,

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|S_n^* f - f\|_\rho = 0.$$

That is, the Szász-Mirakjan-Kantorovich operators S_n^* approximate every function in $C_\rho^k[0,\infty)$ in the weighted norm $\|\cdot\|_\rho$ with respect to P-statistical convergence of order α. Moreover, the rate of approximation is given by

$$\|S_n^* f - f\|_\rho = \mathcal{O}(\omega_\rho(f; n^{-1/2})) \tag{7.5}$$

in the $\mathcal{S}_P^{\alpha}$-sense. This means that the approximation error is controlled by the weighted modulus of continuity evaluated at $n^{-1/2}$.

P r o o f. Let $f \in C_\rho^k[0,\infty)$. From Lemma 7.1 we have

$$\begin{aligned} \|S_n^* e_0 - e_0\|_\rho &= 0, \\ \|S_n^* e_1 - e_1\|_\rho &\leqslant \frac{1}{2n}, \end{aligned}$$

and

$$\|S_n^* e_2 - e_2\|_\rho \leqslant \frac{1}{n} + \frac{1}{3n^2}.$$

Since

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \frac{1}{n} = 0,$$

it follows that

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|S_n^* e_1 - e_1\|_\rho = 0$$

and

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|S_n^* e_2 - e_2\|_\rho = 0.$$

Also,

$$\|S_n^* e_0 - e_0\|_\rho = 0$$

for every n.

Therefore,

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|S_n^* e_j - e_j\|_\rho = 0, \quad j = 0, 1, 2.$$

Now applying Theorem 5.1, we obtain

$$\mathcal{S}_{P^-}^{\alpha} \lim_{n\to\infty} \|S_n^* f - f\|_\rho = 0$$

for every $f \in C_\rho^k[0, \infty)$.

Next, we prove the rate of approximation. From Lemma 7.1,

$$S_n^*((u - x)^2; x) = \frac{x}{n} + \frac{1}{3n^2}.$$

Hence,

$$S_n^*((u - x)^2; x) = \mathcal{O}\left(\frac{1}{n}\right).$$

Using Theorem 6.1, we have

$$\|S_n^* f - f\|_\rho \leqslant C\omega_\rho\left(f; \sqrt{\|S_n^*\varphi^2 - \varphi^2\|_\rho + \Lambda_n}\right),$$

where $C > 0$ is a constant.

Since the moments of S_n^* are of order $1/n$, we obtain

$$\|S_n^*\varphi^2 - \varphi^2\|_\rho + \Lambda_n = \mathcal{O}\left(\frac{1}{n}\right).$$

Therefore,

$$\sqrt{\|S_n^*\varphi^2 - \varphi^2\|_\rho + \Lambda_n} = \mathcal{O}(n^{-1/2}).$$

Substituting this into the above estimate gives

$$\|S_n^* f - f\|_\rho = \mathcal{O}(\omega_\rho(f; n^{-1/2}))$$

in the $\mathcal{S}_P^{\alpha}$-sense. This completes the proof. □

Remark 7.1. The condition

$$\mathcal{S}^{\alpha}_{P^-}\lim_{n\to\infty}\frac{1}{n}=0$$

is important, especially when $\alpha<1$.

For the Abel method, where

$$p_n=1\quad\text{and}\quad R=1,$$

we have

$$\frac{1}{[p(t)]^{\alpha}}\sum_{n\geqslant 0}\frac{1}{n+1}t^n\sim(1-t)^{\alpha}\log\frac{1}{1-t}\to 0\quad\text{as } t\to 1^-.$$

Therefore, the condition holds for every

$$\alpha\in(0,1].$$

Similarly, for the Borel method,

$$\frac{1}{[p(t)]^{\alpha}}\sum_{n\geqslant 0}\frac{p_nt^n}{n}\sim t^{-1}(1-\mathrm{e}^{-t})\mathrm{e}^{-(\alpha-1)t}\to 0.$$

Hence, the required condition is also satisfied for the Borel method.

In view of Theorem 7.1, we introduce the power-series-averaged Szász-Mirakjan-Kantorovich operator.

Define the operators

$$\mathcal{T}_t(f;x):=\frac{1}{p(t)}\sum_{n=0}^{\infty}p_nt^nS_n^*(f;x),\quad t\in(0,R),\ x\geqslant 0. \tag{7.6}$$

For each fixed t, the operator $\mathcal{T}_t$ is a positive linear operator on $C^k_{\rho}[0,\infty)$.

Using Theorem 5.1 together with Lemma 7.1, we obtain

$$\|\mathcal{T}_te_j-e_j\|_{\rho}\leqslant\frac{1}{p(t)}\sum_{n\geqslant 1}p_nt^n\|S_n^*e_j-e_j\|_{\rho},\quad j=0,1,2.$$

If

$$\mathcal{S}^{\alpha}_{P^-}\lim_{n\to\infty}\frac{1}{n}=0,$$

then the right-hand side converges to 0 as

$$t\to R^-.$$

Therefore,

$$\|\mathcal{T}_t e_j - e_j\|_\rho \to 0, \quad j = 0, 1, 2.$$

Hence, the operators $\mathcal{T}_t$ approximate the test functions

$$1, \quad x, \quad x^2$$

in the weighted norm.

Next, we test the proposed operator using the oscillating and decaying function

$$f(x) = \mathrm{e}^{-x/3} \sin(2x) + \frac{1}{2}, \quad x \in [0, 8].$$

The function belongs to $C_\rho^k[0, \infty)$ because it is bounded and

$$\frac{f(x)}{\rho(x)} \to 0 \quad \text{as } x \to \infty.$$

We compute the approximations on a uniform grid of 101 points using:

- ▷ the Szász-Mirakjan operators $S_{20}(f)$ and $S_{50}(f)$,
- ▷ the Szász-Mirakjan-Kantorovich operator $S_{30}^*(f)$,
- ▷ the proposed power-series averaged operator $\mathcal{T}_{40}(f)$ with Borel weights

$$p_n = \frac{1}{n!}.$$

Next, we draw a graph using the MATLAB code for showing the compuational and geometrical behaviour of Szász-Mirakjan-Kantorovich.

The graph shows how well the different operators approximate the target function

$$f(x) = \mathrm{e}^{-x/3} \sin(2x) + \frac{1}{2}$$

on the interval $[0, 8]$.

From Figure 1 we observe that all operators follow the oscillating behavior of the function quite well. As the parameter n increases, the approximation becomes more accurate. In particular, the operator S_{50} gives a better approximation than S_{20} because larger values of n reduce the approximation error.

The Szász-Mirakjan-Kantorovich operator S_{30}^* also approximates the function effectively. Since this operator uses integration over small intervals, it produces a smoother approximation curve.

The proposed power-series averaged operator $\mathcal{T}_{40}$ shows stable and accurate approximation behavior. The averaging process improves the convergence and reduces oscillatory error effects. Geometrically, the graph of $\mathcal{T}_{40}(f)$ stays very close to the graph of the original function on the whole interval.

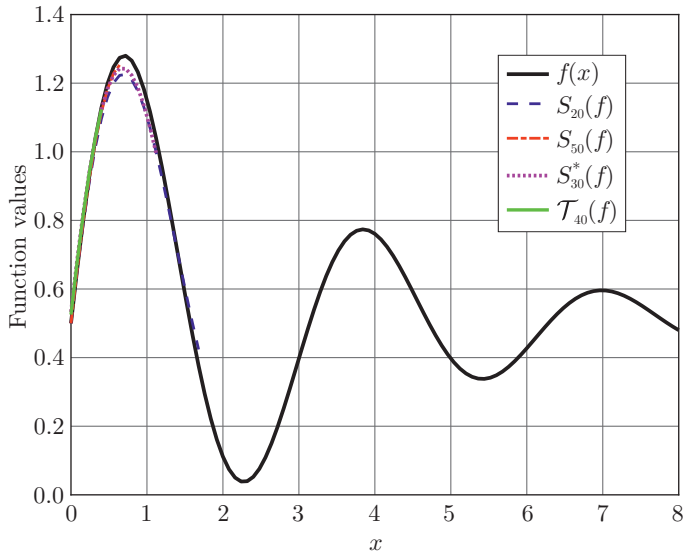

Figure 1. Approximation by Szász-Mirakjan-Kantorovich type operators.

The numerical error table also confirms these observations. The values of both

$$\|\cdot\|_\infty \quad \text{and} \quad \|\cdot\|_\rho$$

decrease as the approximation improves. Smaller error values mean that the operator curve is closer to the exact function curve.

Therefore, the graph and the computed errors together verify the theoretical results proved earlier, namely that the operators converge to the target function and provide effective approximation in the weighted space.

8. Conclusion

In this paper, we introduced the concept of P-statistical convergence of order α and studied its basic properties. We also compared this new type of convergence with other known summability and convergence methods. In addition, we proved a Korovkin-type approximation theorem in the weighted function space $C_\rho[0,\infty)$.

We then applied the theory to the Szász-Mirakjan-Kantorovich operators. Using the weighted modulus of continuity, we obtained the approximation rate

$$\omega_\rho(f; n^{-1/2})$$

in the $\mathcal{S}_P^\alpha$-sense. This shows how fast the operators converge to the target function.

A numerical example was also presented for an oscillating and decaying function on the unbounded interval $[0,\infty)$. The graph and error computations showed that the proposed operators give smooth and accurate approximations. The numerical results also support the theoretical results proved in this work.

There are several possible directions for future research.

(i) One may replace the weight

$$\rho(x) = 1 + x^2$$

by more general weights such as

$$\rho(x) = 1 + x^{2m}, \quad m \in \mathbb{N},$$

or by Orlicz-type weights, and study the approximation properties in the corresponding weighted spaces.

(ii) The present results may also be extended to the q-analogues of the Szász-Mirakjan-Kantorovich operators by combining the theory with q-calculus.

(iii) Another interesting problem is to establish a Voronovskaya-type asymptotic formula for the operators S_n^* in the $\mathcal{S}_P^\alpha$-sense, including higher-order approximation terms.

(iv) It would also be useful to develop a multivariable version of the $\mathcal{S}_P^\alpha$-Korovkin theorem for tensor product Szász-Mirakjan operators on $[0,\infty)^d$.

(v) Finally, the averaged operator $\mathcal{T}_t$ may be applied to practical numerical problems such as inverse problems and density estimation, where its smoothing property can be useful.

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CERTAIN OPERATOR SOLUTIONS TO FERMAT'S TYPE EQUATION ON BERGMAN SPACES

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Abstract. This paper aims to investigate certain operator solutions to Fermat's type equation on Bergman spaces. The solutions have been obtained through Toeplitz operators and Little Hankel operators. In addition, operator matrix solutions in the vector valued Bergman space are examined.

1. **Introduction.** Fermat's last theorem states that there are no non zero integers $x,\ y,\ z\ and\ n$ such that $x^n + y^n = z^n$, where $n > 2$. For more than three centuries, mathematicians have struggled with the unresolved mystery of Fermat's last theorem. In the course of attempting to prove it, number theory underwent significant growth, and Fermat's last theorem eventually rose to become one of the most well-known open problems in mathematics. There have been major efforts to find an elementary proof for this theorem, and numerous particular circumstances, such as the case where $n = 4$ via infinite descent, have been taken into consideration or Legendre and Dirichlet's proof for the case of $n = 5$ and analogous proofs for the other special instances came out one after another. These efforts persisted until Andrew Wiles' well-known proof for general case using elliptical curves and modern mathematics [24]. Researchers who are familiar with fundamental algebraic number theory will find Ernst Eduard Kummer's partial proof of Fermat's last theorem interesting. There are various alternative formulations of Fermat's last theorem that are mathematically equivalent to the original formulation.

In this paper, our goal is to examine conditions under which certain operators defined on Bergman spaces satisfy Fermat's type equation. In order to explain our main result, we mention about Bergman spaces, vector valued Bergman spaces and operators defined on them.

The Bergman space $A^2(\mathbb{D})$ is a Hilbert space consisting of all analytic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_{A^2(\mathbb{D})} = (\int_{\mathbb{D}} |f(z)|^2 dA(z))^{\frac{1}{2}} < \infty,$$

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where $dA(z)$ is the Lebesgue area measure on the open unit disk $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ equals 1. For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

Let $\overline{A^2(\mathbb{D})}$ be the space of conjugate analytic functions in $L^2(\mathbb{D}, dA)$. Then $\overline{A^2(\mathbb{D})} = \{\overline{g} : g \in A^2(\mathbb{D})\}$ is closed in $L^2(\mathbb{D}, dA)$. Let $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator $h_\phi : A^2(\mathbb{D}) \to \overline{A^2(\mathbb{D})}$ be defined by $h_\phi f = \overline{P}(\phi f), f \in A^2(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{A^2(\mathbb{D})}$. There are also numerous equivalent ways of defining little Hankel operators on $A^2(\mathbb{D})$. For illustration, define the map $S_\phi : A^2(\mathbb{D}) \to A^2(\mathbb{D})$ by $S_\phi f = PJ(\phi f)$, where J is selfadjoint, unitary mapping from $L^2(\mathbb{D}, dA)$ into itself given by $Jh(z) = h(\overline{z})$. Observe that, $JS_\phi = h_\phi$. So S_ϕ is unitarily equivalent to h_ϕ.

Let H be an infinite dimensional Hilbert space, then an operator T in $\mathcal{L}(H)$ has a polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry (with $ker(U) = ker(T)$ and $ker(U^*) = ker(T^*)$). The Aluthge transformation $\Delta(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ was first introduced by Aluthge [1]. An operator T is normaloid if $\|T\| = sup\{|\langle Tx, x\rangle|; \|x\| = 1\}$. The Berezin transform of a bounded linear operator T on $A^2(\mathbb{D})$ denoted by $\widetilde{T}$ is defined as

$$\widetilde{T}(w) = \langle Tk_w, k_w\rangle, \text{ for } w \in \mathbb{D}.$$

1.1. **Preliminaries.** Kummer's special case of Fermat's Last Theorem is an especially lovely illustration of how to apply contemporary algebraic number theory to a classical integer problem.

Theorem 1.1. [18] *Let q be a regular (odd) prime. Then $a^q + b^q = c^q$ has no solutions with $a, b, c \in Z$ and a, b, c are prime to q.*

The study of Fermat's last theorem is still quite popular among mathematicians ([2], [5], [8]). In 2021, Nag [23] unveiled an elementary proof Fermat's theorem for Epsilon. In 2022, Mouanda [21] explores the galaxy of sequence of triples of positive integer solutions to the equation $x^2+y^2 = z^2$. In 1966, Domaity [9] proved that the equation $X^4 + Y^4 = Z^4$ can be solved in $M_2(\mathbb{Z})$. In 1993, Frejman [12] explained the solvability of the equation $X^n + Y^n = Z^n$ in the set of positive integer powers of a matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Followings are some studies focusing on the Fermat's last theorem considering integers' matrices.

Theorem 1.2. [20] *Let $\alpha \in \mathbb{N}$ and consider*

$$A_\alpha = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 \end{bmatrix}$$

be a square matrix of coefficient α. The equation $a^4+b^4 = c^4$ has an infinite number of solutions in $\{A_\alpha : \alpha \in \mathbb{N}\}$. However, the equation $a^n + b^n = c^n$, $n \geq 3$, has no solutions in $\{A_{\alpha^2} : \alpha \in \mathbb{N}\}$.

Theorem 1.3. [13] *Let $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$ be a Fibonacci sequence and let F_i, F_j, and F_K be the Fibonacci matrices. If $n \geq 2$ then the equation $F_i^n + F_j^n = F_k^n$ does not hold.*

Theorem 1.4. [19] *The equation $A^n + B^n = C^n$ has solutions (A, B, C, n), where A, B and $C \in \mathbb{X} = \{X^k : k \in \mathbb{N}\}$ if and only if the matrix X is a nilpotent matrix and Tr(X)= det(X)=1.*

In 2004, Jarvis and Meekin [15] studied the Fermat's equation over $\mathbb{Q}(\sqrt{2})$. In 2021, Chein and Meng [4] studied the solvability of Fermat's matrix equation $X^n + Y^n = Z^n$ of 2×2 matrices for $n \geq 4$. They establish that the Fermat's matrix equation has nontrivial solutions in certain 2×2 symmetric rational matrices and $m \times m$ complex row stochastic matrices with row sums belonging to the quadratic field $\mathbb{Q}(\sqrt{2})$. Additionally, they show that the equation has infinitely many solutions in a commuting family of 2×2 symmetric positive semi definite integral matrices.

Motivated by the above works, we explored how the two operators Toeplitz operator and little Hankel operator satisfy Fermat's type equation on the Bergman space. The paper is organized as follows. In section-2, we study the Toeplitz operators satisfying Fermat's type equation and in section-3, we explore the results through little Hankel operators. In section-4, we work with operator matrix solutions to Fermat's type equation.

2. Toeplitz operator.

For $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $A^2(\mathbb{D})$ defined by $T_\phi f = P(\phi f)$; here P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $A^2(\mathbb{D})$.

The Bergman space $A^2(\mathbb{D})$ is a Hilbert space consisting of all analytic functions f on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ for which

$$\|f\|_{A^2(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^2 dA(z)\right)^{\frac{1}{2}} < \infty.$$

After looking over a wide range of literature about Fermat's last theorem we observed that there is no integer solution, rational solution of the Fermat's last theorem but there are few solutions of the Fermat's equation over integer matrices as well as rational fields under certain conditions. In this section, we discuss the operator solutions to Fermat's type equation through Toeplitz operators on the Bergman space.

Theorem 2.1. *Let ϕ, ψ, χ and $\eta \in H^\infty(\mathbb{D})$ and η be a harmonic function with $\|\eta\|_\infty = 1$. If $\phi^n + \psi^n = \chi^n$, where $\widetilde{T_\eta^n}(f) = 1$, $\forall n \in \mathbb{N}$, then*

$$(T_{\eta\phi\overline{\eta}})^n + (T_{\eta\psi\overline{\eta}})^n = (T_{\eta\chi\overline{\eta}})^n$$

Proof. Let $\eta \in h^\infty(\mathbb{D})$ and $\|\eta\|_\infty = 1$. Then $\|T_\eta\| = \|\eta\|_\infty = 1$[6]. Since $\widetilde{T_\eta^n}(f) = 1$ we have, $\langle T_\eta^n k_f, k_f\rangle = 1 = \langle k_f, k_f\rangle$, $\forall f \in \mathbb{D}$. Hence, $T_\eta^n = I$, that implies the T_η is invertible. Then T_η and T_η^{-1} are power bounded and therefore similar to unitary operator [17]. Hence $\|T_\eta\| = 1$ implies that T_η is unitarily equivalent to unitary operator. Therefore, T_η is unitary. That implies $T_\eta T_{\overline{\eta}} = I$. Thus $(T_\eta T_\phi T_{\overline{\eta}})^n = (T_\eta T_{\phi^n} T_{\overline{\eta}})$ and $(T_\eta T_\psi T_{\overline{\eta}})^n = T_\eta T_{\psi^n} T_{\overline{\eta}}$. Therefore,

$$\begin{aligned}(T_\eta T_\phi T_{\overline{\eta}})^n + (T_\eta T_\psi T_{\overline{\eta}})^n &= T_\eta T_{\phi^n} T_{\overline{\eta}} + T_\eta T_{\psi^n} T_{\overline{\eta}} \\ &= T_\eta T_{\phi^n + \psi^n} T_{\overline{\eta}} \\ &= T_\eta T_{\chi^n} T_{\overline{\eta}} = (T_\eta T_\chi T_{\overline{\eta}})^n\end{aligned}$$

That implies,

$$(T_{\eta\phi\overline{\eta}})^n + (T_{\eta\psi\overline{\eta}})^n = (T_{\eta\chi\overline{\eta}})^n.$$

□

Corollary 2.2. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and η, η_1 and $\eta_2 \in L^\infty(\mathbb{D})$ with $T_\eta = T_{\eta_1} + iT_{\eta_2}$ be the cartesian decomposition of T_η. If $\phi^n + \psi^n = \chi^n$, $\forall n \in \mathbb{N}$ and $T_{\eta_1}T_{\eta_2} = T_{\eta_2}T_{\eta_1}$ and $T^2_{\eta_1 - 1} = T_{1-\eta_2}T_{1+\eta_2}$. Then $(T_{1-\eta}T_\phi T_{1-\overline{\eta}})^n + (T_{1-\eta}T_\psi T_{1-\overline{\eta}})^n = (T_{1-\eta}T_\chi T_{1-\overline{\eta}})^n$ $\forall n \in \mathbb{N}$.*

Proof. Given η, η_1 and $\eta_2 \in L^\infty(\mathbb{D})$ and $T_{\eta_1}T_{\eta_2} = T_{\eta_2}T_{\eta_1}$. Now $T_\eta T_{\overline{\eta}} = T_{\overline{\eta}}T_\eta = T^2_{\eta_1} + T^2_{\eta_2}$. Again, since, $T^2_{\eta_1 - 1} = T_{1-\eta_2}T_{1+\eta_2}$ then, $(T_{\eta_1} - I)^2 = (I - T_{\eta_2})(I + T_{\eta_2})$. So $T_\eta T_{\overline{\eta}} = T_{\overline{\eta}}T_\eta = T_{\eta+\overline{\eta}}$. Hence $T_{1-\overline{\eta}}T_{1-\eta} = I = T_{1-\eta}T_{1-\overline{\eta}}$. Therefore, $T_{1-\eta}$ is unitary. Now,

$$\begin{aligned}(T_{1-\eta}T_\phi T_{1-\overline{\eta}})^n + (T_{1-\eta}T_\psi T_{1-\overline{\eta}})^n &= (T_{1-\eta}T^n_\phi T_{1-\overline{\eta}}) + (T_{1-\eta}T^n_\psi T_{1-\overline{\eta}}) \\ &= T_{1-\eta}T_{\phi^n}T_{1-\overline{\eta}} + T_{1-\eta}T_{\psi^n}T_{1-\overline{\eta}} \\ &= T_{1-\eta}T_{\phi^n+\psi^n}T_{1-\overline{\eta}} \\ &= T_{1-\eta}T_{\chi^n}T_{1-\overline{\eta}} = (T_{1-\eta}T_\chi T_{1-\overline{\eta}})^n\end{aligned}$$

□

Corollary 2.3. *Let $\phi_j \in H^\infty(\mathbb{D})$ for $j = 1, 2, \cdots, n$ and $\eta \in h^\infty(\mathbb{D})$ with $\|\eta\|_\infty = 1$. If $\sum_{j=1}^n \phi_j^n = \chi^n$ and $\widetilde{T^n_\eta}(f) = 1$, then $\sum_{j=1}^n (T_\eta T_{\phi_j} T_{\overline{\eta}})^n = (T_\eta T_\chi T_{\overline{\eta}})^n$, $\forall n \in \mathbb{N}$.*

Proof. Let $\eta \in h^\infty(\mathbb{D})$ and $\|\eta\|_\infty = 1$. Then by the Theorem 2.1, T_η is unitary. That implies $T_\eta T_{\overline{\eta}} = I$. Now $(T_{\eta\phi\overline{\eta}})^2 = T_{\eta\phi^2\overline{\eta}}$. Similarly, $(T_\eta T_{\phi_j} T_{\overline{\eta}})^n = (T_\eta T_{\phi_j^n} T_{\overline{\eta}})$. Therefore,

$$\begin{aligned}&(T_\eta T_{\phi_1} T_{\overline{\eta}})^n + (T_\eta T_{\phi_2} T_{\overline{\eta}})^n + \cdots + (T_\eta T_{\phi_n} T_{\overline{\eta}})^n \\ &= (T_\eta T^n_{\phi_1} T_{\overline{\eta}}) + (T_\eta T^n_{\phi_2} T_{\overline{\eta}}) + \cdots + (T_\eta T^n_{\phi_n} T_{\overline{\eta}}) \\ &= T_\eta T_{\phi_1^n} T_{\overline{\eta}} + T_\eta T_{\phi_2^n} T_{\overline{\eta}} + \cdots + T_\eta T_{\phi_n^n} T_{\overline{\eta}} \\ &= T_\eta T_{\phi_1^n + \phi_2^n + \cdots + \phi_n^n} T_{\overline{\eta}} \\ &= T_\eta T_{\chi^n} T_{\overline{\eta}} = (T_\eta T_\chi T_{\overline{\eta}})^n \ (since \sum_{j=1}^n \phi_j^n = \chi^n)\end{aligned}$$

That implies, $\sum_{j=1}^n (T_\eta T_{\phi_j} T_{\overline{\eta}})^n = (T_\eta T_\chi T_{\overline{\eta}})^n$, $\forall n \in \mathbb{N}$. □

Theorem 2.4. *Assume that $\phi, \psi, \chi \in H^\infty(\mathbb{D})$ and $\eta, \xi \in L^\infty(\mathbb{D})$ with $\|\xi\|_\infty \leq 1$. If $\phi^n + \psi^n = \chi^n$, and $\eta = (1 - \frac{1}{n})\xi + \frac{1}{n}$, $n > 2$ and $n \in \mathbb{N}$, then*

$$(T_\eta \Delta(T_\phi) T_{\overline{\eta}})^n + (T_\eta \Delta(T_\psi) T_{\overline{\eta}})^n = (T_\eta \Delta(T_\chi) T_{\overline{\eta}})^n, \forall n \in \mathbb{N}.$$

Proof. Given $\|\xi\|_\infty \leq 1$, $\|I - T_{\frac{1+\xi}{2}}\| = \|I - \frac{I}{2} - \frac{T_\xi}{2}\| \leq \frac{1+\|\xi\|_\infty}{2} < 1$. So, $T_{\frac{1+\xi}{2}}$ is invertible. Therefore, $T_{\frac{1+\xi}{2}}$ can be represented in its polar form, $T_{\frac{1+\xi}{2}} = WQ$, where Q is a positive operator and W is a unitary operator. Hence, $T_{\frac{1+\xi}{2}} = \frac{W(Q_1+Q_2)}{2}$, where $Q_1 = Q + i\sqrt{I - Q^2}$ and $Q_2 = Q - i\sqrt{I - Q^2}$ are unitary operators. So, $T_{1+\xi} = WQ_1 + WQ_2$. Similarly if W_1 , W_2 ... W_{n-1} and U_1 , U_2 ... $U_{n-1} (= W_n)$ such that

$$\begin{aligned}T_{1+\xi} &= W_1 + U_1 \\ T_{1+2\xi} &= W_1 + U_1 + T_\xi.\end{aligned}$$

Therefore,

$$T_{1+(n-1)\xi} = W_1 + U_1 + T_{(n-2)\xi}$$

$$\begin{aligned}&= W_1 + W_2 + U_2 + T_{(n-3)\xi}\\ &= ...\\ &= W_1 + W_2 + ... + (W_n = W_{n-1}).\end{aligned}$$

Then,

$$\begin{aligned}T_\eta &= T_{(1-\frac{1}{n})\xi+\frac{1}{n}}\\ &= \frac{1}{n}T_{1+(n-1)\xi}\\ &= \frac{1}{n}(W_1 + W_2 + ... + W_n).\end{aligned}$$

Therefore T_η is unitary. Now,

$$\begin{aligned}(T_\eta\Delta(T_\phi)T_{\overline{\eta}})^n + (T_\eta\Delta(T_\psi)T_{\overline{\eta}})^n &= (T_\eta\Delta^n(T_\phi)T_{\overline{\eta}}) + (T_\eta\Delta^n(T_\psi)T_{\overline{\eta}})\\ &= T_\eta(\Delta^n(T_\phi) + \Delta^n(T_\psi))T_{\overline{\eta}}\\ &= T_\eta\Delta^n(T_\chi)T_{\overline{\eta}} = (T_\eta\Delta(T_\chi)T_{\overline{\eta}})^n\end{aligned}$$

□

Theorem 2.5. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, and $A, B \in \mathcal{L}(A^2(\mathbb{D}))$ such that $BA = I$, then $(AT_\phi B)^n + (AT_\psi B)^n = (AT_\chi B)^n,\ \forall n \in \mathbb{N}$.*

Proof. Now, $(AT_\phi B)^2 = (AT_\phi B)(AT_\phi B) = (AT_\phi^2 B)$ as $BA = I$. Similarly, $(AT_\phi B)^n = (AT_{\phi^n} B)$. Thus,$(AT_\psi B)^n = AT_{\psi^n} B$

$$\begin{aligned}Therefore, (AT_\phi B)^n + (AT_\psi B)^n &= (AT_\phi^n B) + (AT_\psi^n B)\\ &= AT_{\phi^n} + T_{\psi^n} B\\ &= AT_{\phi^n+\psi^n} B\\ &= AT_{\chi^n} B = (AT_\chi B)^n.\end{aligned}$$

□

Theorem 2.6. *Let ϕ, ψ, χ, η and $\xi \in H^\infty(\mathbb{D})$. Let $T_\eta, T_\xi \in \mathcal{L}(A^2(\mathbb{D}))$ such that $T_\eta X T_\xi = X$ implies $T_{\overline{\eta}} X T_{\overline{\xi}} = X$, where X is self adjoint and has a dense range. If $\phi^n + \psi^n = \chi^n$, and $0 \notin W(X)$ with $[T_\eta, X] = 0$ and $[T_\eta^m, X^m] = 0$, then*

$$(T_{\eta\phi\xi})^n + (T_{\eta\psi\xi})^n = (T_{\eta\chi\xi})^n,\ \forall n \in \mathbb{N}.$$

Proof. Given that $T_\eta X T_\xi = X$ implies $T_{\overline{\eta}} X T_{\overline{\xi}} = X$. That implies $[T_\eta, X^*X] = 0 = [T_\xi, X^*X]$, since X is a self adjoint operator, $X^* = X \Rightarrow X^*X = X^2$ Therefore, $[T_\eta, X^2] = 0 = [T_\xi, X^2]$. Since, $[T_\eta, X] = 0 = [T_\xi, X]$. So, by taking the adjoint and by [22], we have

$$(T_{\overline{\eta}} X T_{\overline{\xi}})^* = (X)^* \Rightarrow T_{\overline{\xi}}^* X^* T_{\overline{\eta}}^* = X^* \Rightarrow T_\xi X T_\eta = X. \tag{1}$$

Since $[T_\eta, X] = 0$, we have $T_\xi T_\eta X = X$. This implies $(T_\xi T_\eta - I)X = 0$ As given X has dense range,

$$(T_\xi T_\eta - I) = 0 \Leftrightarrow T_\xi T_\eta = I.$$

Hence $(T_\eta T_\phi T_\xi)^2 = (T_\eta T_\phi T_\xi)(T_\eta T_\phi T_\xi) = T_\eta T_\phi^2 T_\xi$. Similarly, $(T_\eta T_\phi T_\xi)^n = T_\eta T_\phi^n T_\xi$ and $(T_\eta T_\psi T_\xi)^n = T_\eta T_\psi^n T_\xi$. Therefore,

$$(T_{\eta\phi\xi})^n + (T_{\eta\psi\xi})^n = (T_{\eta\chi\xi})^n,\ \forall n \in \mathbb{N}.$$

□

3. **Little Hankel operator.** Let $\phi \in L^\infty(\mathbb{D})$, the little Hankel operator $h_\phi : A^2(\mathbb{D}) \to \overline{A^2(\mathbb{D})}$ be defined by $h_\phi f = \overline{P}(\phi f), f \in A^2(\mathbb{D})$ where $\overline{P}$ is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $\overline{A^2(\mathbb{D})}$. In this section, we discuss solutions to Fermat's type equation through little Hankel operators.

Theorem 3.1. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and $\eta \in L^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, and h_η is bounded below and power bounded with $\|h_\eta^2\|^2 + 1 = 2\|h_\eta\|^2$, then*

$$(h_\eta T_\phi h_\eta^*)^n + (h_\eta T_\psi h_\eta^*)^n = (h_\eta T_\chi h_\eta^*)^n.$$

Proof. Since given

$$\|h_\eta^2\|^2 + 1 = 2\|h_\eta\|^2. \tag{2}$$

By using method of induction we can show that h_η^n will satisfy the equation (2). Now equation (2) is true for $n = 1$. Let us assume that equation (2) is true for $n = k$, that is $\|h_\eta^{2k}\|^2 + 1 = 2\|h_\eta^k\|^2$, which imply

$$h_\eta^{*^{2k}} h_\eta^{2k} + I = 2h_\eta^{*^k} h_\eta^k. \tag{3}$$

Now we have to show that equation (2) is ture for $n = k + 1$. Now

$$\begin{aligned} h_\eta^{*^{2(k+1)}} h_\eta^{2(k+1)} + I - 2h_\eta^{*^{k+1}} h_\eta^{k+1} &= h_\eta^{*^2}(h_\eta^{*^{2k}} h_\eta^{2k}) h_\eta^2 + I - 2h_\eta^{*^{k+1}} h_\eta^{k+1} \\ &= 2h_\eta^{*^k} h_\eta^k - 2h_\eta^{*^{k-1}} h_\eta^{k-1} - h_\eta^{*^2} h_\eta^2 + I \ (by(3)) \end{aligned}$$

continuing in this way

$$\begin{aligned} &\vdots \\ &= 2h_\eta^{*^2} h_\eta^2 - 2h_\eta^* h_\eta - h_\eta^{*^2} h_\eta^2 + I \\ &= h_\eta^{*^2} h_\eta^2 - 2h_\eta^* h_\eta + I \\ &= 0 \end{aligned}$$

Hence h_η^n satisfies equation (2) therefore,, $\|h_\eta^{2n}\|^2 + 1 = 2\|h_\eta^n\|^2$. Therefore,

$$h_\eta^{*^{2n}} h_\eta^{2n} - 2h_\eta^{*^n} h_\eta^n + I = 0. \tag{4}$$

Again from the equation(2),

$$\|h_\eta^2\|^2 = 2\|h_\eta\|^2 - 1.$$

So

$$\begin{aligned} \|h_\eta^4\|^2 &= 2\|h_\eta^2\|^2 - 1 \\ &= 2(2\|h_\eta\|^2 - 1) - 1 \\ &= 4\|h_\eta\|^2 - 3. \end{aligned}$$

Similarly,

$$\begin{aligned} \|h_\eta^8\|^2 &= 2(4\|h_\eta\|^2 - 3) - 1 \\ &= 8\|h_\eta\|^2 - 7. \end{aligned}$$

In generally, we get

$$\|h_\eta^{2^n}\|^2 = 2^n\|h_\eta\|^2 - (2^n - 1) \tag{5}$$

for every $n \in \mathbb{N}$. Again, given h_η is power bounded, so $\exists$ a positive real number M such that

$$\|h_\eta^n\| \leq M \ for \ n = 1, 2, 3... \ . \tag{6}$$

Therefore, from equations (5) and (6), $M^2 \geq \|h_\eta^{2^n}\|^2 = 2^n\|h_\eta\|^2 - (2^n - 1)$. This implies, $\frac{M^2}{2^n} \geq \|h_\eta\|^2 - 1 + 2^{-n} \geq 0$. As $n \to \infty$, we get $\|h_\eta\|^2 = 1$. Hence,

$$h_\eta^* h_\eta = I. \tag{7}$$

This implies, $ker(h_\eta) = \{0\}$. Therefore, $ker(h_\eta^*) = \{0\}$ ([7] , Lemma 2.1). But we known that $\overline{Range(h_\eta)} = ker(h_\eta^*)^\perp = \{0\}^\perp = A^2(\mathbb{D})$. That implies $Range(h_\eta)$ is dense in $A^2(\mathbb{D})$. Since h_η as bounded below and has dance range, then by [10], h_η is invertible and $Range(h_\eta) = A^2(\mathbb{D})$. Hence, h_η can be written as $h_\eta = VN$, where N is a normal operator and V is the unitary operator. Therefore,

$$h_\eta h_\eta^* = V h_\eta^* h_\eta V^* = I. \tag{8}$$

So, h_η is unitary. Now,

$$\begin{aligned}(h_\eta T_\phi h_\eta^*)^n + (h_\eta T_\psi h_\eta^*)^n &= (h_\eta T_\phi^n h_\eta^*) + (h_\eta T_\psi^n h_\eta^*)\\ &= h_\eta T_{\phi^n} h_\eta^* + h_\eta T_{\psi^n} h_\eta^*\\ &= h_\eta T_{\phi^n + \psi^n} h_\eta^*\\ &= h_\eta T_{\chi^n} h_\eta^* = (h_\eta T_\chi h_\eta^*)^n.\end{aligned}$$

□

Corollary 3.2. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and $\eta \in L^\infty(\mathbb{D})$. If $\phi^n + \psi^n = \chi^n$, $\forall n \in \mathbb{N}$ and $\|h_\eta^2\|^2 + 1 = 2\|h_\eta\|^2$ with $\widetilde{h_\eta^2}(f) = 1$, then*

$$(h_\eta T_\phi h_\eta^*)^n + (h_\eta T_\psi h_\eta^*)^n = (h_\eta T_\chi h_\eta^*)^n, \ \forall n \in \mathbb{N}.$$

Proof. Given that $\widetilde{h_\eta^2}(f) = 1$. Then $\langle h_\eta^2 k_f, k_f\rangle = 1 = \|k_f\|^2$. This implies $\langle h_\eta^2 k_f, k_f\rangle = \langle k_f, k_f\rangle$. That is $h_\eta^2 = I$. It is easy to verify that $h_\eta^2 = I$ if and only if $h_\eta^{*^2} = I$. Since $\|h_\eta^2\|^2 + 1 - 2\|h_\eta\|^2 = 0$ then $h_\eta^{*^2} h_\eta^2 + I - 2h_\eta^* h_\eta = 0$ and an easy computation demonstrates $2I - 2h_\eta^* h_\eta = 0$ or $I = h_\eta^* h_\eta$. Thus h_η is isometry. Further , Since $h_\eta^2 = I$ then h_η is right invertible as well as left invertible and hence h_η is invertible. Therefore, the corollary follows from Theorem 3.1. □

Theorem 3.3. *Let ϕ_1, ϕ_2 and $\chi \in H^\infty(\mathbb{D})$ and $\eta \in L^\infty(\mathbb{D})$. Let h_η is normaloid and $\widetilde{h_\eta^n}(f) = 1$. If $\phi_1^n + \phi_2^n = \chi^n$, then*

$$(h_\eta T_{\phi_1} h_\eta^*)^n + (h_\eta T_{\phi_2} h_\eta^*)^n = (h_\eta T_\chi h_\eta^*)^n, \ \forall n \in \mathbb{N}.$$

Proof. Given $\widetilde{h_\eta^n}(f) = 1$. That implies $\langle h_\eta^n, k_f\rangle = 1 = \langle k_f, k_f\rangle$, implies $h_\eta^n = I$. again since h_η^n is normaloid,

$$\|h_\eta^n\| = \|h_\eta\|^n = \|h_\eta\| = I.$$

Thus, h_η is isometry and $h_\eta^2 = I$. Therefore, $h_\eta^* h_\eta = I$. Now

$$\begin{aligned}(h_\eta T_{\phi_1} h_\eta^*)^n + (h_\eta T_{\phi_2} h_\eta^*)^n &= (h_\eta T_{\phi_1}^n h_\eta^*) + (h_\eta T_{\phi_2}^n h_\eta^*)\\ &= h_\eta (T_{\phi_1^n} + T_{\phi_2^n}) h_\eta^*\\ &= h_\eta T_{\chi^n} h_\eta^* = (h_\eta T_\chi h_\eta^*)^n.\end{aligned}$$

□

Corollary 3.4. *Let ϕ_1, ϕ_2 and $\chi \in H^\infty(\mathbb{D})$ and $\eta \in L^\infty(\mathbb{D})$. Let h_η is normaloid and $\widetilde{h_\eta^n}(f) = 1$. If $\phi_1^n + \phi_2^n = \chi^n$, $\forall n \in \mathbb{N}$, then*

$$(h_\eta \Delta(T_{\phi_1}) h_\eta^*)^n + (h_\eta \Delta(T_{\phi_2}) h_\eta^*)^n = (h_\eta \Delta(T_\chi) h_\eta^*)^n, \ \forall n \in \mathbb{N}.$$

Proof. By Theorem 3.3, we have h_η is isometry and $h_\eta^* h_\eta = I$. Therefore,

$$\begin{aligned}(h_\eta \Delta(T_{\phi_1})h_\eta^*)^n + (h_\eta \Delta(T_{\phi_2})h_\eta^*)^n &= (h_\eta \Delta^n(T_{\phi_1})h_\eta^*) + (h_\eta \Delta^n(T_{\phi_2})h_\eta^*) \\ &= h_\eta(\Delta^n(T_{\phi_1}) + \Delta^n(T_{\phi_2}))h_\eta^* \\ &= h_\eta \Delta^n(T_\chi)h_\eta^* = (h_\eta \Delta(T_\chi)h_\eta^*)^n\end{aligned}$$

□

4. **Operator matrices.** Let H_1, H_2, ... H_n be complex Hilbert spaces and $H = \bigoplus_{i=1}^n H_i$ for $i = 1(n)$. In this section, we discuss about the operator solution to Fermat's type equation on the vector valued Bergman space $A^2(\mathbb{D}) \bigoplus A^2(\mathbb{D})$.

Theorem 4.1. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and $\phi^n + (1-\psi)^n = \chi^n$. If $\phi^{n-1}(\phi - 1) - \psi(1-\psi)^{n-1} = 0$ and $f \geq 0 \in L^\infty(\mathbb{D})$ with $\|T_{1+f}\| < 1$ and $\|S_{1+f}\| < 1$, then,*

$$\begin{aligned}&\left[\begin{pmatrix} T_f & H_{\overline{f}}^* \\ 0 & S_f \end{pmatrix}\begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}\begin{pmatrix} T_f^{-1} & -T_f^{-1}H_{\overline{f}}^* S_f^{-1} \\ 0 & S_f^{-1} \end{pmatrix}\right]^n \\ &+ \left[\begin{pmatrix} T_f & H_{\overline{f}}^* \\ 0 & S_f \end{pmatrix}\begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix}\begin{pmatrix} T_f^{-1} & -T_f^{-1}H_{\overline{f}}^* S_f^{-1} \\ 0 & S_f^{-1} \end{pmatrix}\right]^n \\ &= \left[\begin{pmatrix} T_f^{-1} & -T_f^{-1}H_{\overline{f}}^* S_f^{-1} \\ 0 & S_f^{-1} \end{pmatrix}\begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix}\begin{pmatrix} T_f & H_{\overline{f}}^* \\ 0 & S_f \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.\end{aligned}$$

Proof. Let $P = \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}$. Then, $P^n = \begin{pmatrix} T_\phi^n & nT_\phi^{n-1}T_{\phi-1} \\ 0 & T_\phi^n \end{pmatrix}$. Similarly assume $Q = \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix}$, then, $Q^n = \begin{pmatrix} T_{1-\psi}^n & -nT_\psi T_{1-\psi}^{n-1} \\ 0 & T_{1-\psi}^n \end{pmatrix}$. Thus,

$$\begin{aligned}P^n + Q^n &= \begin{pmatrix} T_\phi^n + T_{1-\psi}^n & n(T_\phi^{n-1}T_{\phi-1} - T_\psi T_{1-\psi}^{n-1}) \\ 0 & T_\phi^n + T_{1-\psi}^n \end{pmatrix} \\ &= \begin{pmatrix} T_{\phi^n} + T_{(1-\psi)^n} & n(T_{\phi^{n-1}}T_{\phi-1} - T_\psi T_{1-\psi^{n-1}}) \\ 0 & T_{\phi^n} + T_{(1-\psi)^n} \end{pmatrix} \\ &= \begin{pmatrix} T_{\phi^n + (1-\psi)^n} & n(T_{\phi^{n-1}(\phi-1)} - T_{\psi(1-\psi)^{n-1}}) \\ 0 & T_{\phi^n(1-\psi)^n} \end{pmatrix}.\end{aligned}$$

Since,

$$\left[\begin{pmatrix} T_f & H_{\overline{f}}^* \\ 0 & S_f \end{pmatrix}\right]^{-1} = \begin{pmatrix} T_f^{-1} & -T_f^{-1}H_{\overline{f}}^* S_f^{-1} \\ 0 & S_f^{-1} \end{pmatrix}$$

and let $A = \begin{pmatrix} T_f & H_{\overline{f}}^* \\ 0 & S_f \end{pmatrix}$. Since $\|T_{1+f}\| < 1$ and $\|S_{1+f}\| < 1$, implies T_f *and* S_f are invertible. Therefore,

$$(APA^{-1})^n + (AQA^{-1})^n = AP^nA^{-1} + AQ^nA^{-1} = A(P^n + Q^n)A^{-1}$$

Again, $P^n + Q^n = \begin{pmatrix} T_{\chi^n} & 0 \\ 0 & T_{\chi^n} \end{pmatrix} = \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix}^n = R^n$. Where, $R = \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix}$.

Since given $\phi^n + (1-\psi)^n = \chi^n$ *and* $\phi^{n-1}(\phi-1) - \psi(1-\psi)^{n-1} = 0$, hence,

$$(APA^{-1})^n + (AQA^{-1})^n = AR^nA^{-1} = (ARA^{-1})^n, \ \forall n \in \mathbb{N}.$$

similarly,

$$(A^{-1}PA)^n + (A^{-1}QA)^n = A^{-1}R^nA = (A^{-1}RA)^n, \ \forall n \in \mathbb{N}.$$

□

Corollary 4.2. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and $\phi^n + (1-\psi)^n = \chi^n$. If $\phi^{n-1}(\phi - 1) - \psi(1-\psi)^{n-1} = 0$ and $f > 0 \in L^\infty(\mathbb{D})$ with $T_f X = 1$ for $X \in \mathcal{L}(A^2\mathbb{D})$, then*

$$\left[\begin{pmatrix} T_f^{\frac{1}{2}} & T_f^{\frac{-1}{2}} H_{\bar{f}}^* \\ 0 & I \end{pmatrix} \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix} \begin{pmatrix} T_f^{\frac{-1}{2}} & -T_f^{-1} H_{\bar{f}}^* \\ 0 & I \end{pmatrix}\right]^n$$
$$+ \left[\begin{pmatrix} T_f^{\frac{1}{2}} & T_f^{\frac{-1}{2}} H_{\bar{f}}^* \\ 0 & I \end{pmatrix} \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix} \begin{pmatrix} T_f^{\frac{-1}{2}} & -T_f^{-1} H_{\bar{f}}^* \\ 0 & I \end{pmatrix}\right]^n$$
$$= \left[\begin{pmatrix} T_f^{\frac{1}{2}} & T_f^{\frac{-1}{2}} H_{\bar{f}}^* \\ 0 & I \end{pmatrix} \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix} \begin{pmatrix} T_f^{\frac{-1}{2}} & -T_f^{-1} H_{\bar{f}}^* \\ 0 & I \end{pmatrix}\right]^n$$

Proof. Let $C = \begin{pmatrix} T_f^{\frac{1}{2}} & T_f^{\frac{-1}{2}} H_{\bar{f}}^* \\ 0 & I \end{pmatrix}$, then, $C^{-1} = \begin{pmatrix} T_f^{\frac{-1}{2}} & -T_f^{-1} H_{\bar{f}}^* \\ 0 & I \end{pmatrix}$. Therefore,

$$(CPC^{-1})^n + (CQC^{-1})^n = (CRC^{-1})^n,$$

where P, Q, and R are $P = \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix}$, $Q = \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix}$ and $R = \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix}$ Hence the result follows from Theorem 4.1. □

Corollary 4.3. *Let ϕ, ψ and $\chi \in H^\infty(\mathbb{D})$ and $\phi^n + (1-\psi)^n = \chi^n$. If $\phi^{n-1}(\phi - 1) - \psi(1-\psi)^{n-1} = 0$, then*

: *(i)* $\left[\begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix} \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix}\right]^n$
$+ \left[\begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix} \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix}\right]^n$
$= \left[\begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix} \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$

: *(ii)* $\left[\begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix} \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix} \begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix}\right]^n$
$+ \left[\begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix} \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix} \begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix}\right]^n$
$= \left[\begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix} \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix} \begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$

: *(iii)* $\left[\begin{pmatrix} R_a & 0 \\ 0 & R_a^* \end{pmatrix} \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix} \begin{pmatrix} R_a^* & 0 \\ 0 & R_a \end{pmatrix}\right]^n$
$+ \left[\begin{pmatrix} R_a & 0 \\ 0 & R_a^* \end{pmatrix} \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix} \begin{pmatrix} R_a^* & 0 \\ 0 & R_a \end{pmatrix}\right]^n$
$= \left[\begin{pmatrix} R_a & 0 \\ 0 & R_a^* \end{pmatrix} \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix} \begin{pmatrix} R_a^* & 0 \\ 0 & R_a \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$

: *(iv)* $\left[\begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix} \begin{pmatrix} T_\phi & T_{\phi-1} \\ 0 & T_\phi \end{pmatrix} \begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix}\right]^n$
$+ \left[\begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix} \begin{pmatrix} T_{1-\psi} & -T_\psi \\ 0 & T_{1-\psi} \end{pmatrix} \begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix}\right]^n$
$= \left[\begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix} \begin{pmatrix} T_\chi & 0 \\ 0 & T_\chi \end{pmatrix} \begin{pmatrix} C_a & 0 \\ 0 & C_a \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$

: *(v) Let $T_{\phi_{ij}} \in L(A^2(\mathbb{D}) \bigoplus A^2(\mathbb{D})), \forall i, \ j = 1, \ 2$. If $T_{\phi_{ij}}$ are invertible for $i = j$. Then,*

$$\left[\begin{pmatrix} T_{\phi_{11}} & 0 \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{\phi} & T_{\phi^{-1}} \\ 0 & T_{\phi} \end{pmatrix}\begin{pmatrix} T_{\phi_{11}}^{-1} & 0 \\ -T_{\phi_{22}}^{-1}T_{\phi_{21}}T_{\phi_{11}}^{-1} & T_{\phi_{22}}^{-1} \end{pmatrix}\right]^n$$
$$+\left[\begin{pmatrix} T_{\phi_{11}} & 0 \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{1-\psi} & -T_{\psi} \\ 0 & T_{1-\psi} \end{pmatrix}\begin{pmatrix} T_{\phi_{11}}^{-1} & 0 \\ -T_{\phi_{22}}^{-1}T_{\phi_{21}}T_{\phi_{11}}^{-1} & T_{\phi_{22}}^{-1} \end{pmatrix}\right]^n$$
$$=\left[\begin{pmatrix} T_{\phi_{11}} & 0 \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{\chi} & 0 \\ 0 & T_{\chi} \end{pmatrix}\begin{pmatrix} T_{\phi_{11}}^{-1} & 0 \\ -T_{\phi_{22}}^{-1}T_{\phi_{21}}T_{\phi_{11}}^{-1} & T_{\phi_{22}}^{-1} \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$$

: *(vi) Let* $T_{\phi_{ij}} \in L(A^2(\mathbb{D}) \bigoplus A^2(\mathbb{D})), \forall i,\ j\ = 1,\ 2.$ *If* $T_{\phi_{21}}$ *and* $T_{\phi_{12}}$ *are invertible. Then,*

$$\left[\begin{pmatrix} 0 & T_{\phi_{12}} \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{\phi} & T_{\phi^{-1}} \\ 0 & T_{\phi} \end{pmatrix}\begin{pmatrix} -T_{\phi_{21}}^{-1}T_{\phi_{22}}T_{\phi_{12}}^{-1} & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & 0 \end{pmatrix}\right]^n$$
$$+\left[\begin{pmatrix} 0 & T_{\phi_{12}} \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{1-\psi} & -T_{\psi} \\ 0 & T_{1-\psi} \end{pmatrix}\begin{pmatrix} -T_{\phi_{21}}^{-1}T_{\phi_{22}}T_{\phi_{12}}^{-1} & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & 0 \end{pmatrix}\right]^n$$
$$=\left[\begin{pmatrix} 0 & T_{\phi_{12}} \\ T_{\phi_{21}} & T_{\phi_{22}} \end{pmatrix}\begin{pmatrix} T_{\chi} & 0 \\ 0 & T_{\chi} \end{pmatrix}\begin{pmatrix} -T_{\phi_{21}}^{-1}T_{\phi_{22}}T_{\phi_{12}}^{-1} & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & 0 \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$$

: *(vii)Let* $T_{\phi_{ij}} \in L(A^2(\mathbb{D}) \bigoplus A^2(\mathbb{D})), \forall i,\ j\ = 1,\ 2.$ *If* $T_{\phi_{21}}$ *and* $T_{\phi_{12}}$ *are invertible. Then,*

$$\left[\begin{pmatrix} T_{\phi_{11}} & T_{\phi_{12}} \\ T_{\phi_{21}} & 0 \end{pmatrix}\begin{pmatrix} T_{\phi} & T_{\phi^{-1}} \\ 0 & T_{\phi} \end{pmatrix}\begin{pmatrix} 0 & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & -T_{\phi_{12}}^{-1}T_{\phi_{11}}T_{\phi_{21}}^{-1} \end{pmatrix}\right]^n$$
$$+\left[\begin{pmatrix} T_{\phi_{11}} & T_{\phi_{12}} \\ T_{\phi_{21}} & 0 \end{pmatrix}\begin{pmatrix} T_{1-\psi} & -T_{\psi} \\ 0 & T_{1-\psi} \end{pmatrix}\begin{pmatrix} 0 & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & -T_{\phi_{12}}^{-1}T_{\phi_{11}}T_{\phi_{21}}^{-1} \end{pmatrix}\right]^n$$
$$=\left[\begin{pmatrix} T_{\phi_{11}} & T_{\phi_{12}} \\ T_{\phi_{21}} & 0 \end{pmatrix}\begin{pmatrix} T_{\chi} & 0 \\ 0 & T_{\chi} \end{pmatrix}\begin{pmatrix} 0 & T_{\phi_{21}}^{-1} \\ T_{\phi_{12}}^{-1} & -T_{\phi_{12}}^{-1}T_{\phi_{11}}T_{\phi_{21}}^{-1} \end{pmatrix}\right]^n, \ \forall n \in \mathbb{N}.$$

Proof. **:** (i) Let $C = \begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix}$. Then $C^{-1} = \begin{pmatrix} U_a & 0 \\ 0 & U_a \end{pmatrix}$. Let $P = \begin{pmatrix} T_{\phi} & T_{\phi^{-1}} \\ 0 & T_{\phi} \end{pmatrix}$, $Q = \begin{pmatrix} T_{1-\psi} & -T_{\psi} \\ 0 & T_{1-\psi} \end{pmatrix}$ and $R = \begin{pmatrix} T_{\chi} & 0 \\ 0 & T_{\chi} \end{pmatrix}$.
Hence by Theorem 4.1, we have

$$(CPC^{-1})^n + (CQC^{-1})^n = (CRC^{-1})^n.$$

: (ii) Let $P = \begin{pmatrix} T_{\phi} & T_{\phi^{-1}} \\ 0 & T_{\phi} \end{pmatrix}$, $Q = \begin{pmatrix} T_{1-\psi} & -T_{\psi} \\ 0 & T_{1-\psi} \end{pmatrix}$ and $R = \begin{pmatrix} T_{\chi} & 0 \\ 0 & T_{\chi} \end{pmatrix}$.
Therefore by using Theorem 4.1, we have

$$(CPC^{-1})^n + (CQC^{-1})^n = (CRC^{-1})^n,$$

where $C = \begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix}$ and $C^{-1} = \begin{pmatrix} U_{\phi_a} & 0 \\ 0 & U_{\phi_a} \end{pmatrix}$.
Similarly, $(iii), (iv), (v), (vi)$ *and* (vii) are obvious from Theorem 4.1. □

Theorem 4.4. *Let* $\phi, \psi \in L^{\infty}(\mathbb{D})$ *and* $a \in \mathbb{D}$. *If* $\begin{pmatrix} T_z & T_{\phi} \\ T_{\phi}^* & S_{\phi} \end{pmatrix}^n + \begin{pmatrix} T_{\phi} & U_a \\ 0 & S_{\psi} \end{pmatrix}^n = \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}^n$, *then* $\begin{pmatrix} T_{\phi_a} & T_{\phi o \phi_a}U_a \\ U_a T_{\phi o \phi_a}^* & S_{\phi} \end{pmatrix}^n + \begin{pmatrix} T_{\phi o \phi_a} & I \\ 0 & S_{\psi} \end{pmatrix}^n = \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}^n$

Proof. Since $\begin{pmatrix} T_{\phi_a} & T_{\phi o \phi_a} U_a \\ U_a T^*_{\phi o \phi_a} & S_\phi \end{pmatrix} = \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T_z & T_\phi \\ T^\star_\phi & S_\psi \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} T_{\phi o \phi_a} & I \\ 0 & S_\psi \end{pmatrix} = \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}$. Then

$$\begin{pmatrix} T_{\phi_a} & T_{\phi o \phi_a} U_a \\ U_a T^*_{\phi o \phi_a} & S_\phi \end{pmatrix}^n + \begin{pmatrix} T_{\phi o \phi_a} & I \\ 0 & S_\psi \end{pmatrix}^n$$

$$= \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix} \left[\begin{pmatrix} T_z & T_\phi \\ T^\star_\phi & S_\phi \end{pmatrix}^n + \begin{pmatrix} T_\phi & U_a \\ 0 & S_\psi \end{pmatrix}^n \right] \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}^n$$

$$= \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}^n \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} U_a & 0 \\ 0 & I \end{pmatrix}^n. \qquad \square$$

5. **Conclusion.** While working with Bergman space and vector valued Bergman space, we gave some operator solutions to Fermat's type equation. The findings should be useful in the search for other forms of operator solutions to these types of equations.

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